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A QUASISTATIC EVOLUTION MODEL FOR THE INTERACTION BETWEEN FRACTURE AND DAMAGE

JEAN-FRANÇOIS BABADJIAN

Abstract. This paper is devoted to the investigation of a quasistatic evolution model for a continuum which undergoes damage and possibly fracture. In both cases, the model appears to be ill posed so that it is necessary to introduce a relaxed variational evolution preserving the irreversibility of the process, the minimality at each time, and the energy balance. From a mechanical point of view, it turns out that the material prefers to form microstructures through the creation of fine mixtures between the damaged and healthy parts of the medium. The brutal character of the damage process is then replaced by a progressive one, where the original damage internal variable, *i.e.* the characteristic function of the damaged part, is replaced by the local volume fraction. The analysis rests on a locality property for mixtures which enables to use an alternative formula for the lower semicontinuous envelope of the elastic energy in terms of the G-closure set.

Keywords: Γ-convergence, Relaxation, Homogenization, Free discontinuity problems, Functions of bounded variation, Fracture mechanics, Damage, Rate independent processes.

2010 Mathematics Subject Classification: 49J45, 49Q20, 35B27, 74E30, 35R35, 74A45.

Contents

1. Introduction

Damage in a brittle elastic medium is characterized by the decrease of its elastic properties during a series of loading-unloading tests. In brutal damage, each point of the material is supposed to exhibit only two states: either damaged or undamaged. Thus the natural internal variable describing the damage phenomenon is the characteristic function of the damaged part of the body. When the material is subjected to a time-dependent loading, where all rate dependent effects like viscosity or inertia are neglected, one can consider damage as a rate independent process, and it is relevant to understand how the damage internal variable evolves in time.

A general abstract theory of quasistatic evolution problems for rate independent materials has been introduced in [35], and it is the subject of many applications as *e.g.* in plasticity [13, 16, 37] or fracture [18, 28, 17] (see [36] for an exhaustive literature on the subject). All these models share the same idea of adding a dissipation energy to the potential energy. The dissipation potential depends on the internal variable(s) of the underlying process, and the fundamental property characterizing the rate independency is that it is positively one homogeneous. Then the same methodology applies by first discretizing the time variable, and then letting the time step tend to zero. A first difficulty occurs at the discrete time level about the well posedness of the successive minimization problems which may fail to have solutions. If so, it is then necessary to introduce the relaxed problem leading to a relaxed variational evolution. This kind of problems has been studied in [38] in the more general framework of stability of rate independent processes through Γ-convergence, which is a well suited mode of convergence for static minimization problems (see [15]). More particular analysis have been carried out for the stability of quasistatic crack evolution through homogenization in [32], or through dimensional reduction in [6].

Partial brittle damage is an example where the original discrete evolution is ill posed (see [29, 27]). Indeed, even at the first time step the minimization of the total energy (*i.e.* the sum of potential energy and the dissipation energy) has no solution due to the non convexity of the elastic energy. From a mathematical point of view, we observe that minimizing sequences oscillate so that they are unable to reach a minimizer. From a mechanical point of view, these oscillations stand for microstructures that the material needs to develop for energetic convenience. The relaxation of the original problem consists thus in the creation of mixtures between the weak material (the part of the body which may already be damaged) and the strong one (the undamaged part of the body), and the characteristic function of the damaged part (the original internal variable) is replaced by the local volume fraction which is a function taking its values in the full interval $[0, 1]$. Thus it may happen that some points of the medium are no longer either healthy or damaged, but that they result from a fine mixture between both phases. Our goal here is to state existence results (see Theorems 1.1 and 1.2) for relaxed quasistatic evolution models involving damage (see [29]) and possibly fracture (see [30]) in the framework of nonlinear elasticity, in contrast with [29, 27] where the authors considered linearized elasticity. Our analysis will be close to that of [27]; we also refer to [31] where a threshold based model is considered, by analogy with stress yield criteria used in plasticity or fracture. We also refer to [3, 4] where are performed numerical simulations of the damage quasistatic evolution model introduced in [29] that we are considering here. They use respectively the homogenization and the level set methods, employed for shape optimization problems which are very close to damage.

The previous discussion shows the importance of the theory of homogenization, and in particular mixtures, for the study of the quasistatic damage evolution. In fact, a localization property proved in [7] for energies resulting from a mixture between two arbitrary materials will be instrumental because it will enable us to obtain an alternative formula for the relaxed energy. The local character of G-closure states that, at least for convex energies, any effective energy obtained by an arbitrary mixture between two materials can be locally recovered as the pointwise limit of a sequence of effective energies obtained as a periodic mixture between the same materials with the same local volume fraction. More specifically, if W_1 and W_2 are convex functions being the stored energy densities of two different materials, and χ is a characteristic function, then we define W_χ by $W_\chi(x, \xi) := \chi(x)W_1(\xi) + (1 - \chi(x))W_2(\xi)$, and by $(W_\chi)_{\rm hom}$ its homogenized energy density (see (4.2)). For any

$\theta \in [0,1]$, define the following set

$$P_\theta(W_1,W_2) := \Big\{ f:\ \exists\, \chi \text{ satisfying } \int_Q \chi\, dx = \theta \text{ and } f = (W_\chi)_{\rm hom} \Big\},$$

made of all energy densities obtained by a periodic mixture of W_1 and W_2 in proportions θ and $1-\theta$. We denote by $G_\theta(W_1,W_2)$ the closure of $P_\theta(W_1,W_2)$ for the pointwise convergence. For any $\theta \in L^\infty(\Omega;[0,1])$, let us further introduce $\mathcal{G}_\theta(W_1,W_2)$ as the set of all possible Carathéodory functions f such that there exists a sequence of characteristic functions (χ_k) satisfying $\chi_k \xrightharpoonup{*} \theta$ in $L^\infty(\Omega;[0,1])$, and

$$\int_\Omega f(x,\nabla u)\, dx = \Gamma\text{-}\lim_{k\to+\infty} \int_\Omega W_{\chi_k}(x,\nabla u)\, dx.$$

The localization result proved in [7] states that if W_1 and W_2 have suitable growth and coercivity conditions, then $f \in \mathcal{G}_\theta(W_1,W_2)$ if and only if $f(x,\cdot) \in G_{\theta(x)}(W_1,W_2)$ for a.e. $x \in \Omega$.

In our application, W_1 and W_2 will stand for, respectively, the stored energy densities of the damaged and undamaged parts of the body occupying the open set $\Omega \subset \mathbb{R}^N$ in its reference configuration. Hence we have $W_1 \le W_2$ since damage decreases the rigidity of the structure. We assume that W_1 and $W_2 \in \mathcal{F}(\alpha,\beta,p)$ satisfy suitable p-growth and p-coercivity conditions with $p>1$ (see (2.1)), and that they are uniformly convex (see (2.4)) and of class $\mathcal{C}^1$. We also assume that Ω has a Lipschitz boundary which can be split into the union of two disjoint sets $\partial\Omega = \partial_N\Omega \cup \partial_D\Omega$, where $\partial_D\Omega$ is open in the relative topology of $\partial\Omega$. We suppose that the Neumann part $\partial_N\Omega$ of the boundary is free, while the Dirichlet part $\partial_D\Omega$ is subjected to a time dependent boundary deformation $g(t)$, where $g \in W^{1,1}([0,T];W^{1,p}(\Omega))$. In view of the growth and coercivity properties of the stored energy density, the natural space of all kinematically admissible deformation fields at time t is given by

$$\mathcal{A}(t) := \{v \in W^{1,p}(\Omega) : v = g(t)\ \mathcal{H}^{N-1}\text{-a.e. on } \partial_D\Omega\}.$$

Note that we are considering here scalar valued functions so that, from the point of view of modeling, only the two dimensional case ($N=2$) is meaningful. Indeed, in that case Ω stands for the basis of an infinite cylinder, and the field u is interpreted as the third component of an anti-plane deformation.

Our first main result is the following theorem.

Theorem 1.1. *Let W_1 and $W_2 \in \mathcal{F}(\alpha,\beta,p)$ be two uniformly convex functions of class $\mathcal{C}^1$ such that $W_1 \le W_2$, and let $g \in W^{1,1}([0,T];W^{1,p}(\Omega))$. Then, for every $t \in [0,T]$, there exist $u(t) \in \mathcal{A}(t)$, $\Theta(t) \in L^\infty(\Omega;[0,1])$ and $W(t) \in \mathcal{F}(\Omega,\alpha,\beta,p)$ such that $W(t)(x,\cdot) \in G_{1-\Theta(t)(x)}(W_1,W_2)$ for a.e. $x \in \Omega$, and*

(i) *Irreversibility: the maps $t \mapsto \Theta(t)$ and $t \mapsto W(t)$ are decreasing;*

(ii) *Unilateral minimality: for any $v \in \mathcal{A}(t)$, any $\theta \in L^\infty(\Omega;[0,1])$ and any $W \in \mathcal{F}(\Omega,\alpha,\beta,p)$ such that $W(x,\cdot) \in G_{\theta(x)}(W_1,W(t)(x,\cdot))$ for a.e. $x \in \Omega$, then*

$$\int_\Omega W(t)(x,\nabla u(t))\, dx \le \int_\Omega W(x,\nabla v)\, dx + \kappa \int_\Omega \Theta(t)\theta\, dx;$$

(iii) *Energy balance: the total energy*

$$\mathcal{E}(t) := \int_\Omega W(t)(x,\nabla u(t))\, dx + \kappa \int_\Omega (1-\Theta(t))\, dx$$

is absolutely continuous with respect to t and

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_\Omega DW(\tau)(x,\nabla u(\tau)) \cdot \nabla \dot g(\tau)\, dx\, d\tau.$$

We will also consider a model where the body can undergo both fracture and damage. A first attempt to this problem has been initiated in [22]. We also refer to [19] where the authors study a quasistatic evolution model for the interplay between fracture and plasticity. The presence of cracks inside the structure implies that the deformation u can exhibit discontinuities. Thus u cannot live anymore in a Sobolev space but in a suitable subspace $SBV^p(\Omega)$ of functions of bounded variation. Thus in this case, the space of all kinematically admissible deformation fields is

$$\mathcal{A}(t) := \{v \in SBV^p(\Omega) : v = g(t)\ \mathcal{H}^{N-1}\text{-a.e. on } \partial_D\Omega\},$$

and we have the following existence result.

Theorem 1.2. *Let W_1 and $W_2 \in \mathcal{F}(\alpha,\beta,p)$ be two uniformly convex functions of class $\mathcal{C}^1$ such that $W_1 \leq W_2$, and let $g \in W^{1,1}([0,T]; W^{1,p}(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N \times [0,T])$. Then, for every $t \in [0,T]$ there exist $u(t) \in \mathcal{A}(t)$, $\Theta(t) \in L^\infty(\Omega;[0,1])$, $W(t) \in \mathcal{F}(\Omega,\alpha,\beta,p)$ such that $W(t)(x,\cdot) \in G_{1-\Theta(t)(x)}(W_1,W_2)$ for a.e. $x \in \Omega$, and a countably $\mathcal{H}^{N-1}$-rectifiable set $\Gamma(t) \subset \overline{\Omega}$ satisfying*

(i) *Irreversibility: the maps $t \mapsto \Theta(t)$ and $t \mapsto W(t)$ are decreasing, and $t \mapsto \Gamma(t)$ is increasing;*

(ii) *Unilateral minimality: for any countably $\mathcal{H}^{N-1}$-rectifiable set $K \subset \overline{\Omega}$ such that $\Gamma(t) \tilde{\subset} K$, any $v \in \mathcal{A}(t)$ such that $J_v \tilde{\subset} K$, any $\theta \in L^\infty(\Omega;[0,1])$ and any $W \in \mathcal{F}(\Omega,\alpha,\beta,p)$ such that $W(x,\cdot) \in G_{\theta(x)}(W_1, W(t)(x,\cdot))$ for a.e. $x \in \Omega$, then*

$$\int_\Omega W(t)(x,\nabla u(t))\,dx + \mathcal{H}^{N-1}(\Gamma(t)\setminus \partial_N\Omega) \leq \int_\Omega W(x,\nabla v)\,dx + \mathcal{H}^{N-1}(K\setminus\partial_N\Omega) + \kappa\int_\Omega \Theta(t)\theta\,dx;$$

(iii) *Energy balance: the total energy*

$$\mathcal{E}(t) := \int_\Omega W(t)(x,\nabla u(t))\,dx + \kappa\int_\Omega (1-\Theta(t))\,dx + \mathcal{H}^{N-1}(\Gamma(t)\setminus\partial_N\Omega)$$

is absolutely continuous with respect to t and

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_\Omega DW(\tau)(x,\nabla u(\tau))\cdot\nabla\dot{g}(\tau)\,dx\,d\tau.$$

Let us now briefly comment our assumptions. Indeed, in this study, we make to main hypothesis. The first one is that the densities W_1 and W_2 are restricted to be convex functions. The reason is that the local character of G-closure proved in [7], and used many times here, has only been proved in the convex case. To our knowledge, it is not known yet whether it holds or not in the general quasiconvex case. Moreover, we not only assumed convexity but uniform convexity (defined in (2.4)). This is mainly due to the results of [32] concerning the stability of quasistatic evolution through Γ-convergence which require some technical assumptions on the elastic energy density (see Lemmas 3.1 and 3.2 and Theorem 3.5). We also choose to consider scalar valued functions instead of vector valued ones (which would have been relevant if one wants to consider three-dimensional elasticity). This assumption is due to two facts: first of all, since we are restricted by the convexity of the elastic energy, we think that it is more relevant to consider scalar valued functions for which convexity is a natural notion, in contrast with the vectorial case where the natural notion is rather quasiconvexity. As explained before, since we are unable to consider the quasiconvex case, we think it is more appropriate to deal with scalar valued functions. Moreover, in the study of the damage-fracture evolution problem, the scalar nature of the problem is used through the application of the maximum principle which helps to get compactness for the sequence of deformations in the space $SBV^p(\Omega)$. In the general vectorial case, the maximum principle would not hold anymore, and it would have been necessary to consider a larger space of generalized special functions of bounded variation as in [17]. Note however that our results (Theorems 1.1 and 1.2) could have been proved exactly in the same way in the vectorial case, but still with the same uniform convexity assumption, and upon imposing the deformation to live inside a fixed "box" for the fracture problem in order to avoid a lack of compactness in $SBV^p(\Omega)$.

The paper is organized as follows: in section 2, we introduce some notations about the functional spaces used in the sequel. In section 3, we recall well known facts about Γ-convergence of integral functionals in Sobolev and special functions of bounded variation spaces. We further prove new Helly type results for sequences of functionals depending monotonically on the time variable. Then, section 4 is dedicated to the homogenization of integral functionals with special emphasis on mixtures. In particular, we prove a finer continuity result on the G-closure set with respect to some Hausdorff metric. Section 5 is devoted to the proof of the existence result (Theorem 1.1) for a relaxed model of quasistatic damage evolution. Finally, in section 6 we prove the second existence result (Theorem 1.2) for another relaxed model of quasistatic evolution involving both fracture and damage.

2. Preliminaries

2.1. **Function spaces.** In the sequel, Ω will always stand for a bounded open subset of $\mathbb{R}^N$. We will also denote by $Q := (0,1)^N$ the open unit square of $\mathbb{R}^N$, and by $B_\rho(x)$ the ball of $\mathbb{R}^N$ of center $x \in \mathbb{R}^N$ and radius $\rho > 0$. If $x = 0$, we simply write B_ρ instead of $B_\rho(0)$.

If $p \geq 1$, we use standard notations for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{1,p}(\Omega)$. The Lebesgue measure in $\mathbb{R}^N$ will be denoted by $\mathcal{L}^N$, while $\mathcal{H}^{N-1}$ will stand for the $(N-1)$-dimensional Hausdorff measure. The symbol $⨍_E$ is used for the average $\mathcal{L}^N(E)^{-1}\int_E$. We will use the notation $\widetilde{\subset}$ for the inclusion of sets, up to a set of zero $\mathcal{H}^{N-1}$ measure. The set of all countably $\mathcal{H}^{N-1}$-rectifiable subsets of $\overline{\Omega}$ is denoted by $\mathcal{R}(\overline{\Omega})$. It is known (see [5]) that for any $K \in \mathcal{R}(\overline{\Omega})$, it is possible to define a generalized normal $\nu_K(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in K$.

We define the space $SBV(\Omega)$ of special functions of bounded variation as the space of all functions $u \in L^1(\Omega)$ such that the distributional derivative Du of u can be represented as a vector valued bounded Radon measure of the form

$$Du = \nabla u \mathcal{L}^N + (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u.$$

Here $\nabla u \in L^1(\Omega;\mathbb{R}^N)$ is the approximate gradient of u (the absolutely continuous part of Du with respect to the Lebesque measure $\mathcal{L}^N$), J_u is the jump set of u which is a countably $\mathcal{H}^{N-1}$-rectifiable set on which one can define $\mathcal{H}^{N-1}$-a.e. a generalized normal denoted ν_u, as well as traces $u^\pm$. We refer to [5] for a detailed description of that space (see also [21]). The space $SBV^p(\Omega)$, for $p > 1$, is a subset of $SBV(\Omega)$ made of all functions $u \in SBV(\Omega)$ such that $\nabla u \in L^p(\Omega;\mathbb{R}^N)$ and $\mathcal{H}^{N-1}(J_u \cap \Omega) < +\infty$. One of the main interests of the space $SBV^p(\Omega)$ is that Mumford-Shah like functionals are coercive in that space according to Ambrosio's compactness Theorem (see [5, Theorem 4.8]).

Theorem 2.1. *Let $(u_n) \subset SBV^p(\Omega)$ be a sequence such that*

$$\sup_{n\in\mathbb{N}} \left(\|u_n\|_{L^\infty(\Omega)} + \|\nabla u_n\|_{L^p(\Omega;\mathbb{R}^N)} + \mathcal{H}^{N-1}(J_{u_n}) \right) < +\infty.$$

Then, there exist a subsequence (u_{n_k}) and a function $u \in SBV^p(\Omega)$ such that $u_{n_k} \to u$ in $L^1(\Omega)$, $\nabla u_{n_k} \rightharpoonup \nabla u$ in $L^p(\Omega;\mathbb{R}^N)$, and $\mathcal{H}^{N-1}(J_u) \leq \liminf_{k\to+\infty} \mathcal{H}^{N-1}(J_{u_{n_k}})$.

The previous result suggests to define a notion of "weak convergence" in $SBV^p(\Omega)$.

Definition 2.1. *Let $(u_n) \subset SBV^p(\Omega)$ and $u \in SBV^p(\Omega)$. We say that u_n converges weakly to u in $SBV^p(\Omega)$, and we write $u_n \rightharpoonup u$ in $SBV^p(\Omega)$, if $u_n \to u$ in $L^1(\Omega)$, $\nabla u_n \rightharpoonup \nabla u$ in $L^p(\Omega;\mathbb{R}^N)$, and $\sup_{n\in\mathbb{N}} \mathcal{H}^{N-1}(J_{u_n}) < +\infty$.*

In the sequel we will also consider the family $P(\Omega) := \{u \in SBV(\Omega) : u(x) \in \{0,1\} \text{ for a.e. } x \in \Omega\}$ of all characteristic functions of sets of finite perimeter in Ω.

2.2. **Integrands.** Let $p > 1$ and $0 < \alpha \leq \beta < +\infty$, then we define $\mathcal{F}(\alpha,\beta,p)$ as the set of all continuous functions $f : \mathbb{R}^N \to [0,+\infty)$ satisfying the following growth and coercivity conditions:

$$\alpha|\xi|^p \leq f(\xi) \leq \beta(1+|\xi|^p) \quad \text{for all } \xi \in \mathbb{R}^N. \tag{2.1}$$

When f is convex, it also satisfies the following p-Lipschitz property (see *e.g.* [14] or [25]): there exists $\gamma > 0$ (depending only on β and p) such that

$$|f(\xi_1) - f(\xi_2)| \leq \gamma(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1})|\xi_1 - \xi_2| \quad \text{for all } \xi_1,\ \xi_2 \in \mathbb{R}^N, \tag{2.2}$$

and if f is further of class $\mathcal{C}^1$, then it differential $Df : \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following $(p-1)$-growth condition

$$|Df(\xi)| \leq \gamma(1+|\xi|^{p-1}) \quad \text{for all } \xi \in \mathbb{R}^N. \tag{2.3}$$

We also define $\mathcal{F}(\Omega,\alpha,\beta,p)$ as the set of all Carathéodory functions $f : \Omega \times \mathbb{R}^N \to [0,+\infty)$ such that $f(x,\cdot) \in \mathcal{F}(\alpha,\beta,p)$ for a.e. $x \in \Omega$.

We now define a notion of convexity which will be instrumental in the quasistatic evolution models studied in sections 5 and 6. We say that a function $f : \mathbb{R}^N \to [0,+\infty)$ is uniformly convex if there exists $\nu > 0$ such that

$$f\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{2}f(\xi_1) + \frac{1}{2}f(\xi_2) - \nu|\xi_1-\xi_2|^2(1+|\xi_1|^2+|\xi_2|^2)^{(p-2)/2} \tag{2.4}$$

for every ξ_1 and $\xi_2 \in \mathbb{R}^N$. In [24, Proposition 2.2] and [23, Proposition 2.5], a complete characterization of such functions is given. Indeed, a function $f \in \mathcal{F}(\alpha,\beta,p)$ is uniformly convex if and only if there exist and constant $c > 0$ and a convex function ϕ such that $f(\xi) = c(1+|\xi|^2)^{p/2} + \phi(\xi)$. By standard convex analysis, if f is further of class $\mathcal{C}^1$, then its differential $Df : \mathbb{R}^N \to \mathbb{R}^N$ is monotone, and satisfies

$$(Df(\xi_2) - Df(\xi_1)) \cdot (\xi_2 - \xi_1) \geq \nu' |\xi_1 - \xi_2|^2 (1 + |\xi_1|^2 + |\xi_2|^2)^{(p-2)/2} \tag{2.5}$$

for every ξ_1 and $\xi_2 \in \mathbb{R}^N$, and for some $\nu' > 0$. As an example, the function $\xi \mapsto (1 + |\xi|^2)^{p/2}$ is uniformly convex. Another important example is the function $f(\xi) := |\xi|^p$ (for $p > 1$) which, unfortunately, fails to be uniformly convex. However, it enjoys good properties as strict convexity and

$$(Df(\xi_2) - Df(\xi_1)) \cdot (\xi_2 - \xi_1) \geq \nu' |\xi_1 - \xi_2|^2 (|\xi_1|^2 + |\xi_2|^2)^{(p-2)/2} \quad \text{if } p > 1. \tag{2.6}$$

Moreover, if $p \geq 2$, one has

$$(Df(\xi_2) - Df(\xi_1)) \cdot (\xi_2 - \xi_1) \geq \nu' |\xi_1 - \xi_2|^p. \tag{2.7}$$

These properties are actually enough for the study of the quasistatic evolution models in sections 5 and 6 (see Lemma 5.6).

3. Γ-convergence

In this section, we recall the definition and standard properties of Γ-convergence. We refer to [11, 15] for a detailed description.

3.1. **Definition and topology of Γ-convergence.** We say that a sequence of functionals $F_n : L^p(\Omega) \to [0, +\infty]$ (with $p > 1$) Γ-converges for the strong $L^p(\Omega)$-topology to $F : L^p(\Omega) \to [0, +\infty]$ if the following properties hold:

(i) for every $u \in L^p(\Omega)$ and every sequence $(u_n) \subset L^p(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$, then

$$F(u) \leq \liminf_{n \to +\infty} F_n(u_n);$$

(ii) for every $u \in L^p(\Omega)$, there exists a sequence $(\bar{u}_n) \subset L^p(\Omega)$ such that $\bar{u}_n \to u$ in $L^p(\Omega)$, and

$$F(u) = \lim_{n \to +\infty} F_n(\bar{u}_n).$$

We now recall several facts about the metrizability of Γ-convergence for lower semicontinuous and equi-coercive functionals. For every $u \in L^p(\Omega)$, we denote by

$$\Psi(u) := \begin{cases} \alpha \displaystyle\int_\Omega |\nabla u|^p \, dx & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

and by $\mathcal{S}_\Psi$ the set of all lower semicontinuous functionals $F : L^p(\Omega) \to [0, +\infty]$ such that $F \geq \Psi$ on $L^p(\Omega)$. For every $F \in \mathcal{S}_\Psi$ and $\lambda > 0$, we denote by F^λ the Moreau-Yosida transform of F defined by

$$F^\lambda(u) := \inf_{v \in L^p(\Omega)} \left\{ F(v) + \lambda \|u - v\|^p_{L^p(\Omega)} \right\} \quad \text{for every } u \in L^p(\Omega).$$

Let Φ be an increasing homeomorphism between $[0, +\infty]$ and $[0, 1]$, (u_j) be a dense sequence in $L^p(\Omega)$, and (λ_j) be a sequence of positive real numbers. For every F and $G \in \mathcal{S}_\Psi$, we define

$$\mathbf{d}_\Gamma(F, G) := \sum_{i,j=1}^{+\infty} \frac{1}{2^{i+j}} \left| \Phi(F^{\lambda_j}(u_i)) - \Phi(G^{\lambda_j}(u_i)) \right|.$$

From [15, Proposition 10.21], it is known that $\mathbf{d}_\Gamma$ defines a distance over $\mathcal{S}_\Psi$. Moreover, by [15, Theorem 10.22], the metric space $(\mathcal{S}_\Psi, \mathbf{d}_\Gamma)$ is compact, and a sequence (F_n) in $\mathcal{S}_\Psi$ Γ-converges to a functional $F \in \mathcal{S}_\Psi$ if and only if $\mathbf{d}_\Gamma(F_n, F) \to 0$.

3.2. **Time dependent functionals.** In the sequel we will be interested in time-dependent sequences of functionals $F_n(t) : L^p(\Omega) \to [0, +\infty]$ which are monotone with respect to the time parameter t. We will need a version of Helly's Theorem as stated below.

Theorem 3.1. *Let $F_n : [0,T] \to \mathcal{S}_\Psi$ be a sequence of functionals which is decreasing with respect to t, i.e., for all $u \in L^p(\Omega)$ and $n \in \mathbb{N}$,*

$$F_n(t)(u) \le F_n(s)(u) \quad \text{for every } 0 \le s \le t \le T.$$

Then there exist a subsequence (n_k) and a functional $F : [0,T] \to \mathcal{S}_\Psi$, still decreasing with respect to t, such that $F_{n_k}(t)$ Γ-converges to $F(t)$ for every $t \in [0,T]$.

Proof. We define the total variation of the map $F_n : [0,T] \to \mathcal{S}_\Psi$ by

$$\mathrm{Var}_\Gamma(F_n; [0,T]) := \sup \left\{ \sum_{k=1}^{h-1} \mathbf{d}_\Gamma(F_n(t_k), F_n(t_{k+1})) : 0 = t_1 < t_2 < \ldots < t_h = T,\ h \in \mathbb{N} \right\}.$$

We claim that

$$\sup_{n\in\mathbb{N}} \mathrm{Var}_\Gamma(F_n; [0,T]) < +\infty. \tag{3.1}$$

Indeed, if $k \in \{1, \ldots, h-1\}$, by the monotonicity of $t \mapsto F_n(t)$ and Φ, we infer that

$$\mathbf{d}_\Gamma(F_n(t_k), F_n(t_{k+1})) := \sum_{i,j=1}^{+\infty} \frac{1}{2^{i+j}} \Phi((F_n(t_k))^{\lambda_j}(u_i)) - \Phi((F_n(t_{k+1}))^{\lambda_j}(u_i)).$$

Consequently, summing up for $k = 1, \ldots, h-1$ we deduce that

$$\begin{aligned} \sum_{k=1}^{h-1} \mathbf{d}_\Gamma(F_n(t_k), F_n(t_{k+1})) &= \sum_{i,j=1}^{+\infty} \frac{1}{2^{i+j}} \Phi((F_n(0))^{\lambda_j}(u_i)) - \Phi((F_n(T))^{\lambda_j}(u_i)) \\ &\le \sum_{i,j=1}^{+\infty} \frac{\Phi((F_n(0))^{\lambda_j}(u_i))}{2^{i+j}} \le \sum_{i,j=1}^{+\infty} \frac{1}{2^{i+j}} = 1. \end{aligned}$$

Then taking the supremum over all partitions in the left hand side of the previous inequality leads to the desired bound (3.1) on the total variation. Since the space $(\mathcal{S}_\Psi, \mathbf{d}_\Gamma)$ is compact, it suffices to apply the general result [35, Theorem 3.2] stated below (Theorem 3.2) in a simpler form. □

Theorem 3.2. *Let $(\mathcal{Y}, d)$ be a compact metric space, and let $Y_n : [0,T] \to \mathcal{Y}$ be a sequence with equi-bounded total variation $\mathrm{Var}_d(Y_n; [0,T])$ with respect to the distance d. Then, there exist a subsequence (n_k) and a function $Y : [0,T] \to \mathcal{Y}$ such that $Y_{n_k}(t) \to Y(t)$ in $(\mathcal{Y}, d)$ for every $t \in [0,T]$.*

When $F_n(t)$ is an integral functional of the form

$$F_n(t)(u) := \begin{cases} \displaystyle\int_\Omega f_n(t)(x, \nabla u)\, dx & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise}, \end{cases} \tag{3.2}$$

we get a localized version of Theorem 3.1 which can be obtained by standard localization techniques and integral representation results (see *e.g.* [11, 12]).

Theorem 3.3. *Let $f_n : [0,T] \to \mathcal{F}(\Omega, \alpha, \beta, p)$ be a sequence of functions which are decreasing with respect to t, i.e., for all $n \in \mathbb{N}$, all $\xi \in \mathbb{R}^N$, and a.e. $x \in \Omega$,*

$$f_n(t)(x, \xi) \le f_n(s)(x, \xi) \quad \text{for every } 0 \le s \le t \le T.$$

Assume further that the functions $f_n(t)(x, \cdot)$ are convex. Then there exist a subsequence (n_k) and a function $f : [0,T] \to \mathcal{F}(\Omega, \alpha, \beta, p)$, still decreasing with respect to t, such that $F_{n_k}(t) : L^p(\Omega) \to [0, +\infty]$ defined by (3.2) Γ-converges to $F(t) : L^p(\Omega) \to [0, +\infty]$ given by

$$F(t)(u) := \begin{cases} \displaystyle\int_\Omega f(t)(x, \nabla u)\, dx & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise}, \end{cases}$$

for every $t \in [0,T]$.

3.3. **Some technical results.** We recall some results proved in [32]. The following lemma stated in [32, Lemma 3.4] is a slight generalization of [17, Lemma 4.9] for varying sequences of functionals. It will be instrumental for deriving energy inequalities in sections 5 and 6.

Lemma 3.1. *Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $H_n : X \times \mathbb{R}^N \to \mathbb{R}^N$ be a sequence of Carathéodory functions which satisfy the following properties:*

(i) *there exists a constant $\gamma > 0$ such that*
$$|H_n(x, \xi)| \leq \gamma(1 + |\xi|^{p-1})$$
for every $\xi \in \mathbb{R}^N$ and μ-a.e. $x \in X$;

(ii) *for all $M > 0$ and all sequences (ξ_n) and (ξ'_n) in $\mathbb{R}^N$ such that $|\xi_n| \leq M$, $|\xi'_n| \leq M$ and $|\xi_n - \xi'_n| \to 0$, then*
$$|H_n(x, \xi_n) - H_n(x, \xi'_n)| \to 0 \quad \text{for } \mu\text{-a.e. } x \in X.$$

Assume that (Φ_n) is bounded in $L^p_\mu(X; \mathbb{R}^N)$ and that (Ψ_n) converges strongly to 0 in $L^p_\mu(X; \mathbb{R}^N)$. Then for every $\Phi \in L^p_\mu(X; \mathbb{R}^N)$,
$$\left| \int_X [H_n(x, \Phi_n(x) + \Psi_n(x)) - H_n(x, \Phi_n(x))] \cdot \Phi(x)\, d\mu(x) \right| \to 0.$$

Using Lemma 3.1, the next result has been proved in [32] pp. 442-443 and generalizes in turn [17, Lemma 4.11]. Under technical assumptions, it ensures the weak convergence of the stresses associated to a sequence of weakly converging deformations (in a suitable topology) for which there holds the convergence of the corresponding elastic energies.

Lemma 3.2. *Let f_n and $f \in \mathcal{F}(\Omega, \alpha, \beta, p)$ be Carathéodory functions satisfying*

(i) *$f_n(x, \cdot)$ and $f(x, \cdot)$ are convex and of class $\mathcal{C}^1$ for a.e. $x \in \Omega$;*

(ii) *for all $M > 0$ and all sequences (ξ_n) and (ξ'_n) in $\mathbb{R}^N$ such that $|\xi_n| \leq M$, $|\xi'_n| \leq M$ and $|\xi_n - \xi'_n| \to 0$, then $|Df_n(x, \xi_n) - Df_n(x, \xi'_n)| \to 0$ for a.e. $x \in \Omega$;*

(iii) *the sequence of functionals*
$$W^{1,p}(\Omega) \ni v \mapsto \int_\Omega f_n(x, \nabla v)\, dx$$
Γ-converges in $L^p(\Omega)$ to
$$W^{1,p}(\Omega) \ni v \mapsto \int_\Omega f(x, \nabla v)\, dx.$$

Assume that $(u_n) \subset SBV^p(\Omega)$ is a sequence weakly converging to some u in $SBV^p(\Omega)$, and that
$$\int_\Omega f_n(x, \nabla u_n)\, dx \to \int_\Omega f(x, \nabla u)\, dx.$$
Then $Df_n(\cdot, \nabla u_n(\cdot)) \rightharpoonup Df(\cdot, \nabla u(\cdot))$ weakly in $L^{p'}(\Omega; \mathbb{R}^N)$ where $1/p + 1/p' = 1$.

3.4. **Γ-convergence in SBV and σ-convergence.** We now recall several facts about Γ-convergence of functionals involving bulk and surface energies defined on the space $SBV^p(\Omega)$. The following result is a particular case of [32, Theorem 5.1] (see also [8]).

Theorem 3.4. *Let f_n and $f \in \mathcal{F}(\Omega, \alpha, \beta, p)$ be Carathéodory functions, and let E be a Borel set satisfying $\mathcal{H}^{N-1}(E) < +\infty$. Then the sequence of functionals*
$$W^{1,p}(\Omega) \ni v \mapsto \int_\Omega f_n(x, \nabla v)\, dx$$
Γ-converges in $L^1(\Omega)$ to
$$W^{1,p}(\Omega) \ni v \mapsto \int_\Omega f(x, \nabla v)\, dx$$
if and only if the sequence of functionals
$$SBV^p(\Omega) \ni v \mapsto \int_\Omega f_n(x, \nabla v)\, dx + \mathcal{H}^{N-1}(J_v \setminus E)$$

Γ-converges in $L^1(\Omega)$ to

$$SBV^p(\Omega) \ni v \mapsto \int_\Omega f(x, \nabla v)\, dx + \mathcal{H}^{N-1}(J_v \setminus E).$$

It is also possible to account for boundary conditions (see [32, Lemma 8.1]). In the sequel, Ω' is an open set containing Ω and such that $\Omega' \setminus \overline{\Omega} \neq \emptyset$.

Lemma 3.3. *Under the same assumptions than in Theorem 3.4, if $(g_n) \subset W^{1,p}(\Omega')$ is such that $g_n \to g$ strongly in $W^{1,p}(\Omega')$, then the sequence of functionals $F_n : L^1(\Omega) \to [0, +\infty]$ defined by*

$$F_n(v) := \begin{cases} \displaystyle\int_\Omega f_n(x, \nabla v) + \mathcal{H}^{N-1}(J_v \setminus E) & \text{if } v \in SBV^p(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

Γ-converges in $L^1(\Omega)$ to the functional $F : L^1(\Omega) \to [0, +\infty]$ defined by

$$F(v) := \begin{cases} \displaystyle\int_\Omega f(x, \nabla v) + \mathcal{H}^{N-1}(J_v \setminus E) & \text{if } v \in SBV^p(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

if and only if the sequence of functionals $G_n : L^1(\Omega') \to [0, +\infty]$ defined by

$$G_n(v) := \begin{cases} \displaystyle\int_\Omega f_n(x, \nabla v) + \mathcal{H}^{N-1}(J_v \setminus E) & \text{if } \begin{cases} v \in SBV^p(\Omega'), \\ v = g_n \text{ a.e. in } \Omega' \setminus \overline{\Omega}, \end{cases} \\ +\infty & \text{otherwise,} \end{cases}$$

Γ-converges in $L^1(\Omega')$ to the functional $G : L^1(\Omega') \to [0, +\infty]$ defined by

$$G(v) := \begin{cases} \displaystyle\int_\Omega f(x, \nabla v) + \mathcal{H}^{N-1}(J_v \setminus E) & \text{if } \begin{cases} v \in SBV^p(\Omega'), \\ v = g \text{ a.e. in } \Omega' \setminus \overline{\Omega}, \end{cases} \\ +\infty & \text{otherwise.} \end{cases}$$

We next recall a notion of convergence for countably $\mathcal{H}^{N-1}$-rectifiable sets introduced in [32] called σ-convergence which is related to the Γ-convergence of interfacial energies (see also [17] for another notion of convergence called σ^p-convergence related to the weak convergence in $SBV^p(\Omega)$). Indeed consider a sequence (K_n) in $\mathcal{R}(\overline{\Omega})$, and define for any $u \in L^1(\Omega')$

$$H_n(u) := \begin{cases} \mathcal{H}^{N-1}(J_u \setminus K_n) & \text{if } u \in P(\Omega'), \\ +\infty & \text{otherwise.} \end{cases}$$

Assume that H_n Γ-converges in $L^1(\Omega')$ to $H : L^1(\Omega') \to [0, +\infty]$ defined by

$$H(u) := \begin{cases} \displaystyle\int_{J_u} h(x, \nu_u)\, d\mathcal{H}^{N-1} & \text{if } u \in SBV^p(\Omega'), \\ +\infty & \text{otherwise,} \end{cases}$$

for some function $h : \Omega' \times \mathbb{S}^{N-1} \to [0, +\infty)$.

Definition 3.1. *We say that the sequence (K_n) σ-converges in $\overline{\Omega}$ to some $K \in \mathcal{R}(\overline{\Omega})$ if the functional H_n Γ-converges in $L^1(\Omega')$ to H, and if K is the (unique) set in $\mathcal{R}(\overline{\Omega})$ such that*

$$h(x, \nu_K(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K,$$

and such that for any $K' \in \mathcal{R}(\overline{\Omega})$ we have

$$h(x, \nu_{K'}(x)) = 0 \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in K' \;\Rightarrow\; K' \,\tilde{\subset}\, K.$$

The σ-convergence enjoys good compactness and lower semicontinuity properties as the next result shows (see Propositions 6.3 and 6.7 in [32]).

Proposition 3.1. *Let (K_n) be a sequence in $\mathcal{R}(\overline{\Omega})$ such that*

$$\sup_{n \in \mathbb{N}} \mathcal{H}^{N-1}(K_n) < +\infty.$$

Then, there exist a subsequence (K_{n_k}) and $K \in \mathcal{R}(\overline{\Omega})$ such that K_{n_k} σ-converges to K in $\overline{\Omega}$, and

$$\mathcal{H}^{N-1}(K \setminus E) \leq \liminf_{n \to +\infty} \mathcal{H}^{N-1}(K_n \setminus E)$$

for any Borel set $E \subset \overline{\Omega}$ *with* $\mathcal{H}^{N-1}(E) < +\infty$.

The notion of σ-convergence is relevant to study the stability of unilateral minimizers through Γ-convergence of the energies (see [32, Theorem 7.2]). Indeed, let $f_n \in \mathcal{F}(\Omega, \alpha, \beta, p)$ be a sequence of Carathéodory integrands. We suppose that Ω has a Lipschitz boundary which can be split into the union of two disjoint sets $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, where $\partial_D\Omega$ is open in the relative topology of $\partial\Omega$. In particular, there exists an open subset Ω' of $\mathbb{R}^N$ such that $\Omega \subset \Omega'$ and $\partial_D\Omega = \Omega' \cap \partial_D\Omega$. Then consider three sequences $(g_n) \subset W^{1,p}(\Omega')$, $(u_n) \subset SBV^p(\Omega')$ and $(K_n) \subset \mathcal{R}(\overline{\Omega})$ such that $J_{u_n} \mathrel{\widetilde{\subset}} K_n$ and $u_n = g_n$ a.e. in $\Omega' \setminus \overline{\Omega}$ for each $n \in \mathbb{N}$. We further assume that the pair (u_n, K_n) is a unilateral minimizer with respect to the integrand f_n, *i.e.*,

$$\int_\Omega f_n(x, \nabla u_n)\, dx + \mathcal{H}^{N-1}(K_n \setminus \partial_N\Omega) \le \int_\Omega f_n(x, \nabla v)\, dx + \mathcal{H}^{N-1}(K' \setminus \partial_N\Omega),$$

for any $K' \in \mathcal{R}(\overline{\Omega})$ such that $K_n \mathrel{\widetilde{\subset}} K'$, and any $v \in SBV^p(\Omega')$ satisfying $v = g_n$ a.e. in $\Omega' \setminus \overline{\Omega}$ and $J_v \mathrel{\widetilde{\subset}} K'$.

Theorem 3.5. *Suppose that* $g_n \to g$ *in* $W^{1,p}(\Omega)$, $u_n \rightharpoonup u$ *in* $SBV^p(\Omega')$ *and* K_n σ*-converges to* K *in* $\overline{\Omega}$. *Assume further that* f_n *and* $f \in \mathcal{F}(\Omega, \alpha, \beta, p)$ *are Carathéodory functions satisfying*

(i) $f_n(x, \cdot)$ *and* $f(x, \cdot)$ *are convex and of class* $\mathcal{C}^1$ *for a.e.* $x \in \Omega$;
(ii) *for all* $M > 0$ *and all sequences* (ξ_n) *and* (ξ'_n) *in* $\mathbb{R}^N$ *such that* $|\xi_n| \le M$, $|\xi'_n| \le M$ *and* $|\xi_n - \xi'_n| \to 0$, *then* $|Df_n(x, \xi_n) - Df_n(x, \xi'_n)| \to 0$ *for a.e.* $x \in \Omega$;
(iii) *the sequence of functionals*

$$W^{1,p}(\Omega) \ni v \mapsto \int_\Omega f_n(x, \nabla v)\, dx$$

Γ*-converges in* $L^p(\Omega)$ *to*

$$W^{1,p}(\Omega) \ni v \mapsto \int_\Omega f(x, \nabla v)\, dx.$$

Then (u, K) *is a unilateral minimizer with respect to the integrand* f, *i.e.*,

$$\int_\Omega f(x, \nabla u)\, dx + \mathcal{H}^{N-1}(K \setminus \partial_N\Omega) \le \int_\Omega f(x, \nabla v)\, dx + \mathcal{H}^{N-1}(K' \setminus \partial_N\Omega),$$

for any $K' \in \mathcal{R}(\overline{\Omega})$ *such that* $K \mathrel{\widetilde{\subset}} K'$, *and any* $v \in SBV^p(\Omega')$ *satisfying* $v = g$ *a.e. in* $\Omega' \setminus \overline{\Omega}$ *and* $J_v \mathrel{\widetilde{\subset}} K'$. *Moreover,*

$$\int_\Omega f_n(x, \nabla u_n)\, dx \to \int_\Omega f(x, \nabla u)\, dx.$$

We conclude this section by stating a transfer of jump set result which was initialy proved in [28] and further extended in [17]. The version below is taken from [32, Theorem 7.4].

Theorem 3.6. *Let* f *and* f_n *be as in Theorem 3.5, and consider a sequence* $(K_n) \subset \mathcal{R}(\overline{\Omega})$ *such that*

$$\sup_{n \in \mathbb{N}} \mathcal{H}^{N-1}(K_n) < +\infty,$$

which σ*-converges in* $\overline{\Omega}$ *to some* $K \in \mathcal{R}(\overline{\Omega})$. *For every* $v \in SBV^p(\Omega)$, *there exists a sequence* $(v_n) \subset SBV^p(\Omega)$ *with* $v_n \rightharpoonup v$ *in* $SBV^p(\Omega)$ *such that*

$$\int_\Omega f_n(x, \nabla v_n)\, dx \to \int_\Omega f(x, \nabla v)\, dx,$$

and

$$\limsup_{n \to +\infty} \mathcal{H}^{N-1}(J_{v_n} \setminus (\partial_N\Omega \cup K_n)) \le \mathcal{H}^{N-1}(J_v \setminus (\partial_N\Omega \cup K)).$$

Moreover, if $v \in SBV^p(\Omega')$ *and if there is a sequence* $(g_n) \subset W^{1,p}(\Omega')$ *which converges strongly in* $W^{1,p}(\Omega')$ *to some* $w \in W^{1,p}(\Omega')$ *such that* $v = g$ *a.e. in* $\Omega' \setminus \overline{\Omega}$, *then one can assume that the sequence* $(v_n) \subset SBV^p(\Omega')$ *satisfies* $v_n = g_n$ *a.e. in* $\Omega' \setminus \overline{\Omega}$ *for each* $n \in \mathbb{N}$.

4. Homogenization and G-closure

In this section we recall well known facts about the homogenization of (convex) integral functionals, and then we will specialize our study to the case of mixtures for which local properties of effective materials can be obtained (see [7]).

Given $f \in \mathcal{F}(Q, \alpha, \beta, p)$, we denote by $f_{\rm hom}(\xi)$ its homogenized energy density defined by

$$\begin{aligned} f_{\rm hom}(\xi) &:= \lim_{T\to+\infty} \inf\left\{ \fint_{(0,T)^N} f(\langle y\rangle, \xi + \nabla\varphi(y))\,dy : \varphi \in W^{1,p}_{\rm per}((0,T)^N)\right\} \\ &= \inf_{k\in\mathbb{N}} \inf\left\{ \fint_{(0,k)^N} f(\langle y\rangle, \xi + \nabla\varphi(y))\,dy : \varphi \in W^{1,p}_{\rm per}((0,k)^N)\right\}, \end{aligned} \tag{4.1}$$

where $\langle y\rangle$ denotes the fractional part of the vector $y \in \mathbb{R}^N$. It is known (see [10, 39]) that the functional $F_{\rm hom} : L^p(\Omega) \to [0, +\infty]$ defined by

$$F_{\rm hom}(v) := \begin{cases} \displaystyle\int_\Omega f_{\rm hom}(\nabla v)\,dx & \text{if } v \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

is the Γ-limit in $L^p(\Omega)$ of the family $F_\varepsilon : L^p(\Omega) \to [0, +\infty]$ given by

$$F_\varepsilon(v) := \begin{cases} \displaystyle\int_\Omega f\left(\left\langle \frac{x}{\varepsilon}\right\rangle, \nabla v\right) dx & \text{if } v \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, since the functions v are scalar valued, the asymptotic formula (4.1) reduces to the following single cell formula

$$f_{\rm hom}(\xi) = \inf_{\varphi\in W^{1,p}_{\rm per}(Q)} \int_Q f(y, \xi + \nabla\varphi(y))\,dy. \tag{4.2}$$

We now deal with a particular case of composite materials which are mixtures. Given a characteristic function $\chi \in L^\infty(\Omega; \{0, 1\})$ and two convex functions W_1 and $W_2 \in \mathcal{F}(\alpha, \beta, p)$, we define the Carathéodory integrand $W_\chi \in \mathcal{F}(\Omega, \alpha, \beta, p)$ by

$$W_\chi(x, \xi) := \chi(x)W_1(\xi) + (1 - \chi(x))W_2(\xi).$$

For any $\theta \in [0, 1]$, define the following set

$$\begin{aligned} P_\theta(W_1, W_2) \;:=\; \Big\{ & f \in \mathcal{F}(\alpha, \beta, p) \text{ such that there exists } \chi \in L^\infty(Q; \{0, 1\}) \\ & \text{satisfying } \int_Q \chi\,dx = \theta \text{ and } f = (W_\chi)_{\rm hom}\Big\}, \end{aligned}$$

made of all energy densities obtained by a periodic mixture of W_1 and W_2 in proportions θ and $1-\theta$. We denote by $G_\theta(W_1, W_2) := \overline{P_\theta}(W_1, W_2)$ the closure of $P_\theta(W_1, W_2)$ for the pointwise convergence.

For any $\theta \in L^\infty(\Omega; [0, 1])$, let us further introduce $\mathcal{G}_\theta(W_1, W_2)$ as the set of all possible densities $f \in \mathcal{F}(\Omega, \alpha, \beta, p)$ such that there exists a sequence of characteristic functions $(\chi_k) \subset L^\infty(\Omega; \{0, 1\})$ satisfying $\chi_k \overset{*}{\rightharpoonup} \theta$ in $L^\infty(\Omega; [0, 1])$, and

$$\int_\Omega f(x, \nabla v)\,dx = \Gamma\text{-}\lim_{k\to+\infty} \int_\Omega W_{\chi_k}(x, \nabla v)\,dx.$$

A localization result proved in [7] states that

$$\mathcal{G}_\theta(W_1, W_2) = \Big\{ f \in \mathcal{F}(\Omega, \alpha, \beta, p) : f(x, \cdot) \in G_{\theta(x)}(W_1, W_2) \text{ for a.e. } x \in \Omega\Big\}. \tag{4.3}$$

In other words, in the case of mixtures, periodic homogenization captures locally any kind of homogenization phenomena. Hence mixtures are completely characterized by periodic geometries.

4.1. **Some technical results.** We now state several technical results about mixtures that will be of use in the sequel. The first result is a generalization of [7, Theorem 3.5] which can be proved exactly in the same way by using the Scorza-Dragoni Theorem (see [20, 25]).

Proposition 4.1. *Let f and $g \in \mathcal{F}(\Omega, \alpha, \beta, p)$ be convex functions in the second variable, and $\theta \in L^\infty(\Omega; [0,1])$. Given $W \in \mathcal{F}(\Omega, \alpha, \beta, p)$ such that $W(x, \cdot) \in \mathcal{G}_{\theta(x)}(f(x,\cdot), g(x,\cdot))$ for a.e. $x \in \Omega$, then there exists a sequence (χ_n) in $L^\infty(\Omega; \{0,1\})$ such that $\chi_n \xrightharpoonup{*} \theta$ in $L^\infty(\Omega; [0,1])$, and*

$$\int_\Omega W(x, \nabla v)\, dx = \Gamma\text{-}\lim_{n \to +\infty} \int_\Omega \Big[\chi_n f(x, \nabla v) + (1 - \chi_n) g(x, \nabla v)\Big]\, dx.$$

The second result will also be useful. Its proof is very standard and relies on the Decomposition Lemma [26], and the approximation of Lebesgue measurable sets by open or closed sets.

Lemma 4.1. *Let f_n and $f \in \mathcal{F}(\Omega, \alpha, \beta, p)$ be convex functions in the second variable such that*

$$\int_\Omega f(x, \nabla v)\, dx = \Gamma\text{-}\lim_{n \to +\infty} \int_\Omega f_n(x, \nabla v)\, dx.$$

Then, for every convex function $g \in \mathcal{F}(\alpha, \beta, p)$ and every $\chi \in L^\infty(\Omega; \{0,1\})$, one has

$$\int_\Omega \Big[\chi g(\nabla v) + (1-\chi) f(x, \nabla v)\Big]\, dx = \Gamma\text{-}\lim_{n \to +\infty} \int_\Omega \Big[\chi g(\nabla v) + (1-\chi) f_n(x, \nabla v)\Big]\, dx.$$

The last result states that $\mathcal{C}^1$ regularity of the integrand is preserved by homogenization.

Lemma 4.2. *Let W_1 and $W_2 \in \mathcal{F}(\alpha, \beta, p)$ be convex and $\mathcal{C}^1$ functions, and $\theta \in [0,1]$. If $f \in \mathcal{G}_\theta(W_1, W_2)$, then f is also of class $\mathcal{C}^1$.*

Proof. If $f \in \mathcal{G}_\theta(W_1, W_2)$, then there exists a sequence of characteristic functions $(\chi_n) \subset L^\infty(Q; \{0,1\})$ such that $\chi_n \xrightharpoonup{*} \theta$ in $L^\infty(Q; [0,1])$, and

$$\int_Q f(\nabla v)\, dx = \Gamma\text{-}\lim_{n \to +\infty} \int_Q \big(\chi_n W_1(\nabla v) + (1-\chi_n) W_2(\nabla v)\big)\, dx.$$

Set $W_{\chi_n}(x, \xi) := \chi_n(x) W_1(\xi) + (1 - \chi_n(x)) W_2(\xi)$ for every $(x, \xi) \in Q \times \mathbb{R}^N$. Then $W_{\chi_n} \in \mathcal{F}(Q, \alpha, \beta, p)$ is convex and of class $\mathcal{C}^1$ in the second variable, and for every ξ and $\xi' \in \mathbb{R}^N$, one has

$$|DW_{\chi_n}(x, \xi) - DW_{\chi_n}(x, \xi')| \leq |DW_1(\xi) - DW_1(\xi')| + |DW_2(\xi) - DW_2(\xi')|. \tag{4.4}$$

In particular, if ξ_n and $\xi'_n \in \mathbb{R}^N$ are two sequences in $\mathbb{R}^N$ such that $|\xi_n - \xi'_n| \to 0$ and $|\xi_n| \leq M$ and $|\xi'_n| \leq M$ for some $M > 0$, then $|DW_{\chi_n}(x, \xi_n) - DW_{\chi_n}(x, \xi'_n)| \to 0$ for a.e. $x \in \Omega$. According to [32, Proposition 3.5], we infer that f is of class $\mathcal{C}^1$. □

4.2. **Regularity results.** We next address more precise information on the G-closure set $G_\theta(W_1, W_2)$. We will prove that the map $\theta \mapsto G_\theta(W_1, W_2)$ is continuous with respect to the Hausdorff distance in a suitable metric space. This property will be used in sections 5 and 6 to prove an alternative formula for the lower semicontinuous envelope of the elastic energy in terms of the G-closure set. To do that, we need to prove a regularity result of Meyers type for the solutions of the minimization problem defining the cell formula (4.2).

Proposition 4.2. *Let $f \in \mathcal{F}(Q, \alpha, \beta, p)$ be a convex function in its second variable. There exist a constant $\mathcal{C} > 0$ and an exponent $r > p$ (both depending only on N, p, α and β) such that for any $\xi \in \mathbb{R}^N$, if $\varphi_\xi \in W^{1,p}_{\rm per}(Q)$ is a solution of the cell problem (4.2), then*

$$\left(\int_Q |\nabla \varphi_\xi|^r\, dx\right)^{1/r} \leq \mathcal{C}(1 + |\xi|).$$

Proof. The proof is very closed to that of Theorem 3.1, Chapter V in [33] (see also [34, Theorem 6.7]). It is clear from the growth and coercivity conditions (2.1) that there exists a constant $c > 0$ depending only on p, α and β such that

$$\left(\int_Q |\nabla \varphi_\xi|^p\, dx\right)^{1/p} \leq c(1 + |\xi|). \tag{4.5}$$

Extend φ_ξ by Q-periodicity to $\mathbb{R}^N$, and for all $1 < s < t < 2$ consider a cut-off function $\eta \in \mathcal{C}^\infty_c((0,2)^N;[0,1])$ such that $\eta = 1$ on $(0,s)^N$, $\eta = 0$ outside $(0,t)^N$ and $|\nabla \eta| \leq 2/(t-s)$. Let

$$\overline{\varphi}_\xi := \fint_{(0,2)^N} \varphi_\xi \, dx = \int_Q \varphi_\xi \, dx$$

be the average of φ_ξ over $(0,2)^N$ and define $v := \eta \, \overline{\varphi}_\xi + (1-\eta)\, \varphi_\xi \in W^{1,p}_{\rm per}((0,2)^N)$. In particular, extending $f(\cdot,\xi)$ by Q-periodicity to $\mathbb{R}^N$ and using (4.1), we infer that

$$\fint_{(0,2)^N} f(x, \xi + \nabla \varphi_\xi)\, dx = f_{\rm hom}(\xi) \leq \fint_{(0,2)^N} f(x,\xi + \nabla v)\, dx,$$

and since $v = \varphi_\xi$ outside $(0,t)^N$, it follows that

$$\int_{(0,t)^N} f(x,\xi+\nabla\varphi_\xi)\, dx \leq \int_{(0,t)^N} f(x,\xi+\nabla v)\, dx.$$

Using the growth and coercivity conditions (2.1) and the fact that $s < t$ we obtain that

$$\alpha \int_{(0,s)^N} |\xi + \nabla \varphi_\xi|^p \, dx \leq \beta \int_{(0,t)^N} \big(1 + |\xi + \nabla v|^p\big)\, dx.$$

Hence there exists a constant $c_1 = c_1(\alpha,\beta,p,N) > 0$ such that

$$\int_{(0,s)^N} |\nabla \varphi_\xi|^p \, dx \leq c_1 \left(\int_{(0,t)^N \setminus (0,s)^N} |\nabla \varphi_\xi|^p \, dx + \frac{1}{(t-s)^p} \int_{(0,2)^N} |\varphi_\xi - \overline{\varphi}_\xi|^p \, dx + (1 + |\xi|^p) \right). \quad (4.6)$$

Applying the *"hole filling"* method, we eliminate the first term in the right hand side by summing to (4.6) c_1 times the left hand side of the previous inequality. We obtain

$$\int_{(0,s)^N} |\nabla \varphi_\xi|^p \, dx \leq \frac{c_1}{1+c_1} \int_{(0,t)^N} |\nabla \varphi_\xi|^p \, dx + \frac{1}{(t-s)^p} \int_{(0,2)^N} |\varphi_\xi - \overline{\varphi}_\xi|^p \, dx + (1+|\xi|^p).$$

From Lemma 3.1, Chapter V in [33] (see also [34, Lemma 6.1]), we get that

$$\int_Q |\nabla \varphi_\xi|^p \, dx \leq c_2 \left(\int_{(0,2)^N} |\varphi_\xi - \overline{\varphi}_\xi|^p \, dx + (1 + |\xi|^p) \right)$$

where $c_2 = c_2(\alpha,\beta,p,N) > 0$. Using the Sobolev-Poincaré inequality (see *e.g.* [21, Theorem 4.5-2]), there exists a constant $c_3 = c_3(\alpha,\beta,p,N) > 0$ such that

$$\int_Q |\nabla \varphi_\xi|^p \, dx \leq c_3 \left(\left(\fint_{(0,2)^N} |\nabla \varphi_\xi|^q \, dx \right)^{p/q} + (1 + |\xi|^p) \right)$$

where $q^* = p$, *i.e.* $q = Np/(N+p) < p$. In view of Proposition 1.1, Chapter V in [33] (see also [34, Theorem 6.6]), we obtain the existence of a constant $C > 0$ and an exponent $r > p$ depending only on α, β, p and N such that

$$\left(\int_Q |\nabla \varphi_\xi|^r \, dx \right)^{1/r} \leq C \left(\left(\fint_{(0,2)^N} |\nabla \varphi_\xi|^p \, dx \right)^{1/p} + (1 + |\xi|) \right).$$

The result follows from (4.5) and the Q-periodicity of φ_ξ. $\square$

We now use the previous result to prove several properties of functions of the type $(W_\chi)_{\rm hom}$.

Lemma 4.3. *For every $\xi \in \mathbb{R}^N$ and every χ, $\chi' \in L^\infty(Q;\{0,1\})$, one has*

$$|(W_\chi)_{hom}(\xi) - (W_{\chi'})_{hom}(\xi)| \leq c\,(1+|\xi|^p) \left(\int_Q |\chi - \chi'|^{r/(r-p)} \, dx \right)^{(r-p)/r}, \quad (4.7)$$

where $r > p$ is the same exponent as in Proposition 4.2.

Proof. Let $\varphi \in W^{1,p}_{\rm per}(Q)$ such that

$$\int_Q W_\chi(x, \xi + \nabla\varphi)\, dx = (W_\chi)_{\rm hom}(\xi).$$

By Proposition 4.2, there exist a constant $\mathcal{C} > 0$ and an exponent $r > p$, depending only on α, β, p and N, such that

$$\left(\int_Q |\nabla\varphi|^r\, dx\right)^{1/r} \leq \mathcal{C}(1 + |\xi|).$$

Hence Hölder's inequality and the p-growth condition (2.1) imply that

$$\begin{aligned}
&(W_{\chi'})_{\rm hom}(\xi) - (W_\chi)_{\rm hom}(\xi)\\
&\quad\leq \int_Q W_{\chi'}(x, \xi + \nabla\varphi)\, dx - \int_Q W_\chi(x, \xi + \nabla\varphi)\, dx\\
&\quad\leq \beta \int_Q |\chi' - \chi| \big(1 + |\xi + \nabla\varphi|^p\big)\, dx\\
&\quad\leq \beta \left(\int_Q |\chi' - \chi|^{r/(r-p)}\, dx\right)^{(r-p)/r} \left(1 + \left(\int_Q |\xi + \nabla\varphi|^r\, dx\right)^{p/r}\right)\\
&\quad\leq c\,(1 + |\xi|^p) \left(\int_Q |\chi - \chi'|^{r/(r-p)}\, dx\right)^{(r-p)/r}.
\end{aligned}$$

Relation (4.7) follows from the symmetric roles played by χ and χ'. □

We define the space $\mathcal{E}_p$ of all continuous functions $f : \mathbb{R}^N \to \mathbb{R}$ such that there exists the limit

$$\lim_{|\xi| \to +\infty} \frac{f(\xi)}{1 + |\xi|^{p+1}} = 0.$$

This space is isomorphic to the space $\mathcal{C}_0(\mathbb{R}^N)$ of all continuous functions on $\mathbb{R}^N$ vanishing at infinity, hence it is a separable Banach space for the norm

$$\|f\| := \sup_{\xi \in \mathbb{R}^N} \frac{|f(\xi)|}{1 + |\xi|^{p+1}}.$$

We recall the definition of the Hausdorff distance between two closed sets A and B in $\mathcal{E}_p$:

$$\mathbf{d}_{\mathcal{H}}(A, B) := \max\left\{\sup_{g \in B} \inf_{f \in A} \|f - g\|\ ,\ \sup_{f \in A} \inf_{g \in B} \|f - g\|\right\}.$$

We want to measure the Hausdorff distance between sets of the type $G_\theta(W_1, W_2)$ for varying $\theta \in [0, 1]$. To do that we first need to prove that these sets are actually bounded and closed in $\mathcal{E}_p$.

Lemma 4.4. *For any $\theta \in [0, 1]$ the set $G_\theta(W_1, W_2)$ is a bounded and closed subset of $\mathcal{E}_p$.*

Proof. That $G_\theta(W_1, W_2)$ is a bounded subset of $\mathcal{E}_p$ is a direct consequence of the fact that every functions $f \in G_\theta(W_1, W_2)$ satisfy the uniform p-growth and p-coercivity condition (2.1).

Let us show that $G_\theta(W_1, W_2)$ is closed. Since $\mathcal{E}_p$ is a metric space it is enough to check that $f \in G_\theta(W_1, W_2)$ whenever $f_j \to f$ in $\mathcal{E}_p$ and $f_j \in G_\theta(W_1, W_2)$ for each $j \in \mathbb{N}$. Let $(\chi^j_k)_{k \in \mathbb{N}} \subset L^\infty(Q; \{0, 1\})$ be such that

$$\int_Q \chi^j_k(y)\, dy = \theta \tag{4.8}$$

and $(W_{\chi^j_k})_{\rm hom} \to f_j$ pointwise as $k \to +\infty$. Since $(W_{\chi^j_k})_{\rm hom} \in \mathcal{F}(\alpha, \beta, p)$, by (2.1) and (2.2), the sequence $((W_{\chi^j_k})_{\rm hom})_{k \in \mathbb{N}}$ is locally bounded and uniformly equi-continuous. Hence by the Ascoli Theorem, $(W_{\chi^j_k})_{\rm hom} \to f_j$ locally uniformly as $k \to +\infty$, and consequently, for every $M > 1$,

$$\lim_{j \to +\infty} \lim_{k \to +\infty} \sup_{|\xi| \leq M} \frac{\big|(W_{\chi^j_k})_{\rm hom}(\xi) - f(\xi)\big|}{1 + |\xi|^{p+1}} = 0.$$

On the other hand, from the p-growth condition (2.1) we have that

$$\sup_{|\xi|>M} \frac{\left|(W_{\chi_k^j})_{\rm hom}(\xi) - f(\xi)\right|}{1+|\xi|^{p+1}} \leq 2\beta \sup_{|\xi|>M} \frac{1+|\xi|^p}{1+|\xi|^{p+1}} \leq \frac{c}{1+M} \xrightarrow[M\to+\infty]{} 0$$

uniformly with respect to j and k. As a consequence

$$\lim_{j\to+\infty} \lim_{k\to+\infty} \left\|(W_{\chi_k^j})_{\rm hom} - f\right\| = 0,$$

and by a diagonalization argument, it is possible to find an increasing sequence $k_j \to +\infty$ as $j \to +\infty$ such that $(W_{\chi^j_{k_j}})_{\rm hom} \to f$ in $\mathcal{E}_p$ (and also pointwise). Together with (4.8) it ensures that $f \in G_\theta(W_1, W_2)$. □

The following result is an adaptation of [40, Lemma 4.1] (see also [2, Lemma 2.1.7]) which states that the map $\theta \mapsto G_\theta$ is continuous with respect to the Hausdorff convergence in $\mathcal{E}_p$.

Proposition 4.3. *There exist a constant $c > 0$ and an exponent $r > p$ (depending only on α, β, p and N) such that for every θ_1, $\theta_2 \in [0,1]$*

$$\mathbf{d}_{\mathcal{H}}(G_{\theta_1}(W_1,W_2), G_{\theta_2}(W_1,W_2)) \leq c|\theta_1-\theta_2|^{(r-p)/r}.$$

Proof. Assume without loss of generality that $\theta_1 < \theta_2$. Let $f \in G_{\theta_2}(W_1, W_2)$, there exists a sequence $(\chi_k) \subset L^\infty(Q; \{0,1\})$ such that

$$\int_Q \chi_k \, dx = \theta_2 \tag{4.9}$$

and $(W_{\chi_k})_{\rm hom}(\xi) \to f(\xi)$. Define $E_k := \{\chi_k = 1\}$ and set $\ell_k(\rho) := \mathcal{L}^N(B_\rho \cap E_k)$. The function ℓ_k is continuous and increasing, and it satisfies $\ell_k(0) = 0$ and $\ell_k(\sqrt{N}) = \theta_2$. As a consequence there exists $\rho_k \in (0, \sqrt{N})$ such that $\ell_k(\rho_k) = \theta_1$. We define now $\tilde{E}_k := B_{\rho_k} \cap E_k$ and $\tilde\chi_k := \chi_{\tilde E_k}$. By the Ascoli Theorem, one may extract a subsequence (not relabeled) such that $(W_{\tilde\chi_k})_{\rm hom}$ converges locally uniformly (hence also pointwise) to some continuous fonction $g \in \mathcal{F}(\alpha, \beta, p)$. Moreover, since

$$\int_Q \tilde\chi_k \, dx = \theta_1, \tag{4.10}$$

we deduce that $g \in G_{\theta_1}(W_1, W_2)$.

By construction, we have $\tilde E_k \subset E_k$, and thus $\tilde\chi_k \leq \chi_k$. Consequently by (4.7), (4.9) and (4.10) we obtain that

$$\begin{aligned} |g(\xi) - f(\xi)| &= \lim_{k\to+\infty} \left|(W_{\tilde\chi_k})_{\rm hom}(\xi) - (W_{\chi_k})_{\rm hom}(\xi)\right| \\ &\leq \lim_{k\to+\infty} c\,(1+|\xi|^p) \left(\int_Q (\chi_k - \tilde\chi_k)^{r/(r-p)}\, dx\right)^{(r-p)/r} \\ &\leq c\,(1+|\xi|^p)(\theta_2-\theta_1)^{(r-p)/r}, \end{aligned}$$

and then $\|f - g\| \leq c\,|\theta_1 - \theta_2|^{(r-p)/r}$. Since this estimate holds for a particular $g \in G_{\theta_1}(W_1, W_2)$ and for any arbitrary $f \in G_{\theta_2}(W_1, W_2)$ we deduce that

$$\sup_{f\in G_{\theta_2}} \inf_{g\in G_{\theta_1}} \|f-g\| \leq c\,|\theta_1-\theta_2|^{(r-p)/r}.$$

Similarly one can show that

$$\sup_{g\in G_{\theta_1}} \inf_{f\in G_{\theta_2}} \|f-g\| \leq c\,|\theta_1-\theta_2|^{(r-p)/r},$$

which concludes the proof of the Proposition in view of the definition of the Hausdorff distance. □

5. Quasistatic damage evolution

This section is devoted to the investigation of a model of quasistatic evolution for a continuum that undergoes damage. Specifically, we prove Theorem 1.1.

5.1. **Description of the damage evolution model.** Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary which represents the reference configuration of a nonlinearly elastic material. We assume that at each point of Ω, the stored energy density of this body can take only two values W_1 or $W_2 \in \mathcal{F}(\alpha, \beta, p)$ which are supposed to be uniformly convex functions (see (2.4)) of class $\mathcal{C}^1$. The density W_1 is that of the damaged material while W_2 is that of the undamaged one so that $W_1 \leq W_2$.

Denoting by $\chi : \Omega \to \{0, 1\}$ the characteristic function of the damaged part of Ω, the stored energy density of the material is

$$W_\chi(x, \xi) := \chi(x) W_1(\xi) + (1 - \chi(x)) W_2(\xi).$$

We now describe the boundary conditions. To do that, we split the boundary of Ω into the union of two sets $\partial\Omega = \partial_N\Omega \cup \partial_D\Omega$, where $\partial_D\Omega$ is open in the relative topology of $\partial\Omega$. We suppose that the Neumann part $\partial_N\Omega$ is free, while on the Dirichlet part $\partial_D\Omega$, a time dependent boundary deformation $g(t)$ is imposed. We assume that $g \in W^{1,1}([0, T]; W^{1-1/p,p}(\partial_D\Omega))$ is the trace on $\partial_D\Omega$ of a function still denoted $g \in W^{1,1}([0, T]; W^{1,p}(\Omega))$. In view of the growth and coercivity properties of the stored energy density, the natural space of all kinematically admissible deformation fields at time t is given by

$$\mathcal{A}(t) := \{v \in W^{1,p}(\Omega) : v = g(t)\ \mathcal{H}^{N-1}\text{-a.e. on } \partial_D\Omega\}.$$

Note that the Dirichlet boundary condition is the only driving mechanism which will make evolve the damage process. Here the internal variable which will describe this evolution is the characteristic function χ of the damaged zone. Following [29] we adopt the yield criterion that the deformation gradient ∇u must stay inside the set

$$\mathcal{R} := \{\xi \in \mathbb{R}^N : W_2(\xi) - W_1(\xi) < \kappa\},$$

where $\kappa > 0$ is a constant of the material. Hence at a discrete time level, if $u_i \in \mathcal{A}(t_i)$ denotes the current deformation field, and χ_i is the characteristic function of the damaged part of the material at time t_i, the evolution law for χ_i to pass from time t_i to time t_{i+1} is

$$\chi_{i+1}(x) = \begin{cases} 0 & \text{if} \quad \chi_i(x) = 0 \text{ and } \nabla u_i(x) \in \mathcal{R}, \\ 1 & \text{if} \quad \chi_i(x) = 1 \text{ or } \nabla u_i(x) \notin \mathcal{R}, \end{cases}$$

taking into account the irreversibility of the damage process. Hence the globally dissipated energy from the initial time up to time t_i is given by

$$\kappa \int_\Omega \chi_i(x)\, dx.$$

The evolution of damage as previously described may be formulated by a two fields partial minimization problem upon introducing the functional $\mathcal{L}_i$ defined by

$$\mathcal{L}_i(u, \chi) := \int_\Omega [\chi(x) W_1(\nabla u(x)) + (1 - \chi(x)) W_2(\nabla u(x))]\, dx + \kappa \int_\Omega \chi(x)\, dx$$

for $u \in \mathcal{A}(t_i)$ and $\chi \in \mathcal{X}_i := \{\chi \in L^\infty(\Omega; \{0, 1\}) : \chi \geq \chi_{i-1}\}$. Then one looks for (u_i, χ_i) as a global minimizer of $\mathcal{L}_i$ over $\mathcal{A}(t_i) \times \mathcal{X}_i$. Unfortunately, even at the first time step, this minimization may fail to have solutions and it is necessary to relax the original problem. The mechanical reason for this phenomenon is that, for energetic convenience, the material prefers to create microstructures. Thus the brittle character of the damage process is loosed for a progressive evolution where the characteristic function of the damaged part is replaced by the volume fraction which takes its values in the full interval $[0, 1]$ (instead of $\{0, 1\}$). Hence, at a given point x of Ω, the effective material will not be exclusively either healthy or damaged anymore, but it can also result from a mixture between the weak and strong materials leading to a composite material.

5.2. **Time discretization.** Let $0 = t_0^k < t_1^k < \ldots < t_{n(k)}^k = T$ be a discretization of the time interval $[0, T]$ such that

$$\max_{1 \leq i \leq n(k)} (t_i^k - t_{i-1}^k) \to 0, \tag{5.1}$$

as $k \to +\infty$. We define $g_i^k := g(t_i^k) \in W^{1,p}(\Omega)$.

5.2.1. *First time step.* At the initial time, $g_0^k = g(0)$, and one wants to minimize

$$(v,\chi) \mapsto \int_\Omega \Big[\chi W_1(\nabla v) + (1-\chi) W_2(\nabla v) + \kappa\chi\Big]\,dx,$$

among all $(v,\chi) \in \mathcal{A}(0) \times L^\infty(\Omega;\{0,1\})$. Minimizing first with respect to χ leads to the following nonconvex integrand

$$\psi_0(\xi) := \min\{W_1(\xi)+\kappa, W_2(\xi)\},$$

and the previous minimization problem is equivalent to

$$I_0 := \inf_{v\in\mathcal{A}(0)} \int_\Omega \psi_0(\nabla v)\,dx. \tag{5.2}$$

The lack of convexity of the integrand ψ_0 may prevent (5.2) to have solutions, so that it is necessary to compute the relaxed problem. It is well known (see [25]) that it suffices to replace ψ_0 by its convexification. In the scalar case, it also coincides with its quasiconvexification $Q\psi_0$ defined by

$$Q\psi_0(\xi) := \inf_{\varphi\in W^{1,p}_{\rm per}(Q)} \int_Q \psi_0(\xi+\nabla\varphi)\,dx,$$

and

$$I_0 = \min_{v\in\mathcal{A}(0)} \int_\Omega Q\psi_0(\nabla v)\,dx. \tag{5.3}$$

There is another way to express $Q\psi_0$ in terms of G-closure. In [22] (see [27] in the case of linearized elasticity and [29]), it is stated that the quasiconvexification of ψ_0 can be expressed in terms of the sets $P_\theta(W_1,W_2)$ or $G_\theta(W_1,W_2)$. Namely,

Lemma 5.1. *For every $\xi \in \mathbb{R}^N$,*

$$\begin{aligned} Q\psi_0(\xi) &= \inf_{\theta\in[0,1]} \left[\inf_{W^*\in P_\theta(W_1,W_2)} W^*(\xi) + \kappa\theta\right] \\ &= \min_{\theta\in[0,1]} \left[\min_{W^*\in G_\theta(W_1,W_2)} W^*(\xi) + \kappa\theta\right]. \end{aligned}$$

Proof. To prove the first equality, we remark that for any $\theta\in[0,1]$,

$$\inf_{W^*\in P_\theta(W_1,W_2)} W^*(\xi)+\kappa\theta = \inf_{\{\chi\in L^\infty(Q;\{0,1\}):\int_Q \chi\,dy=\theta\}} \inf_{\varphi\in W^{1,p}_{\rm per}(Q)} \int_Q \big(W_\chi(y,\xi+\nabla\varphi(y)) + \kappa\chi(y)\big)\,dy$$

so that, inverting the orders of infimum,

$$\begin{aligned} \inf_{\theta\in[0,1]} &\left[\inf_{W^*\in P_\theta(W_1,W_2)} W^*(\xi)+\kappa\theta\right] \\ &= \inf_{\varphi\in W^{1,p}_{\rm per}(Q)} \inf_{\chi\in L^\infty(Q;\{0,1\})} \int_Q \big(W_\chi(y,\xi+\nabla\varphi(y)) + \kappa\chi(y)\big)\,dy \\ &= \inf_{\varphi\in W^{1,p}_{\rm per}(Q)} \int_Q \psi_0(\xi+\nabla\phi(y))\,dy = Q\psi_0(\xi). \end{aligned}$$

To prove the second equality, let $\theta_k\in[0,1]$ and $f_k \in P_{\theta_k}(W_1,W_2)$ be minimizing sequences such that $f_k(\xi)+\kappa\theta_k \to Q\psi_0(\xi)$ as $k\to+\infty$. Up to a subsequence, there is no loss of generality to assume that $\theta_k\to\theta$. Moreover, since $f_k \in \mathcal{F}(\alpha,\beta,p)$, using an argument similar to that of the proof of Lemma 4.4 based on the Ascoli Theorem, one can assume that f_k converges to some W^* in $\mathcal{E}_p$. Moreover, by Proposition 4.3, we deduce that $G_{\theta_k}(W_1,W_2)$ converges in the sense of Hausdorff to $G_\theta(W_1,W_2)$, and thus $W^*\in G_\theta(W_1,W_2)$ and $W^*(\xi)+\kappa\theta = Q\psi_0(\xi)$. □

Let $u_0\in\mathcal{A}(0)$ be a minimizer of (5.3), and let $\theta_0(x)\in[0,1]$ and $W_0(x,\cdot)\in G_{\theta_0(x)}(W_1,W_2)$ be such that

$$Q\psi_0(\nabla u_0(x)) = W_0(x,\nabla u_0(x)) + \kappa\theta_0(x)$$

for a.e. $x\in\Omega$. The next result asserts that it is possible to select θ_0 and W_0 as measurable functions of x.

Lemma 5.2. *The functions θ_0 and W_0 can be chosen to be respectively measurable and Carathéodory. Consequently $\theta_0 \in L^\infty(\Omega;[0,1])$, $W_0 \in \mathcal{F}(\Omega,\alpha,\beta,p)$ and $W_0(x,\cdot)$ is uniformly convex and of class $\mathcal{C}^1$ for a.e. $x \in \Omega$.*

Proof. Since $W_0(x,\cdot) \in G_{\theta_0(x)}(W_1,W_2)$, it follows that it is a convex function, and according to Lemma 4.2 it is also of class $\mathcal{C}^1$.

We now prove the measurability properties of θ_0 and $W_0(\cdot,\xi)$. Assume first that ∇u_0 is a simple measurable function, then θ_0 and $W_0(\cdot,\xi)$ are simple measurable functions as well. In the general case, there exists a sequence of simple measurable functions ξ_n which pointwise a.e. converges to ∇u_0. Let $\theta_n(x)$ and $f_n(x,\xi)$ be simple measurable functions of x such that for each $n \in \mathbb{N}$, $f_n(x,\cdot) \in G_{\theta_n(x)}(W_1,W_2)$ and

$$Q\psi_0(\xi_n(x)) = f_n(x,\xi_n(x)) + \kappa\theta_n(x)$$

for a.e. $x \in \Omega$. Define

$$W_0(x,\xi) := \limsup_{n\to+\infty} f_n(x,\xi) \quad \text{and} \quad \theta_0(x) := \limsup_{n\to+\infty} \theta_n(x)$$

which are consequently measurable functions of x as well. For fixed $x \in \Omega$, extract a suitable subsequence (possibly depending on x) such that

$$W_0(x,\xi) := \lim_{j\to+\infty} f_{n_j}(x,\xi) \quad \text{and} \quad \theta_0(x) := \lim_{j\to+\infty} \theta_{n_j}(x).$$

Arguing exactly as in the proof of Lemma 4.4, we can show that $f_{n_j}(x,\cdot) \to W_0(x,\cdot)$ in $\mathcal{E}_p$. On the other hand, since $f_{n_j}(x,\cdot) \in G_{\theta_{n_j}(x)}(W_1,W_2)$ and $\theta_{n_j}(x) \to \theta_0(x)$, we infer thanks to Proposition 4.3 that $W_0(x,\cdot) \in G_{\theta_0(x)}(W_1,W_2)$. Then, by the continuity of $Q\psi_0$ we get that

$$\begin{aligned} Q\psi_0(\nabla u_0(x)) &= \lim_{j\to+\infty} Q\psi_0(\xi_{n_j}(x)) = \lim_{j\to+\infty} \left\{ f_{n_j}(x,\xi_{n_j}(x)) + \kappa\theta_{n_j}(x) \right\} \\ &= W_0(x,\nabla u_0(x)) + \kappa\theta_0(x). \end{aligned}$$

To show that it is uniformly convex, we first note that thanks to the local character of G-closure (4.3), then $W_0(x,\cdot) \in \mathcal{G}_{\theta_0(x)}(W_1,W_1)$ for a.e. $x \in \Omega$. Now fix such a point $x \in \Omega$, and consider ξ_1 and $\xi_2 \in \mathbb{R}^N$. There exist a sequence of characteristic functions $(\chi_n) \subset L^\infty(Q;\{0,1\})$ such that $\chi_n \overset{*}{\rightharpoonup} \theta_0(x)$ in $L^\infty(Q;[0,1])$, and sequences (φ_n), $(\phi_n) \subset W^{1,p}(Q)$ weakly converging to 0 in $W^{1,p}(Q)$ satisfying

$$W_0(x,\xi_1) = \lim_{n\to+\infty} \int_Q \left[\chi_n(y)W_1(\xi_1+\nabla\varphi_n(y)) + (1-\chi_n(y))W_2(\xi_1+\nabla\varphi_n(y))\right] dy,$$

and

$$W_0(x,\xi_2) = \lim_{n\to+\infty} \int_Q \left[\chi_n(y)W_1(\xi_2+\nabla\phi_n(y)) + (1-\chi_n(y))W_2(\xi_2+\nabla\phi_n(y))\right] dy.$$

Note that from the p-growth and p-coercivity conditions (2.1), we have that

$$\limsup_{n\to+\infty} \|\nabla\varphi_n\|^p_{L^p(Q;\mathbb{R}^N)} \le C(1+|\xi_1|^p), \quad \limsup_{n\to+\infty} \|\nabla\phi_n\|^p_{L^p(Q;\mathbb{R}^N)} \le C(1+|\xi_2|^p), \tag{5.4}$$

for some constant $C > 0$ depending only on α, β and p. Define $\psi_n = (\varphi_n+\phi_n)/2 \in W^{1,p}(Q)$ with $\psi_n \rightharpoonup 0$ in $W^{1,p}(Q)$ so that

$$\begin{aligned} W_0\left(x,\frac{\xi_1+\xi_2}{2}\right) \le \liminf_{n\to+\infty} \int_Q \Bigg[\chi_n(y)W_1\left(\frac{\xi_1+\xi_2}{2}+\nabla\psi_n(y)\right) \\ +(1-\chi_n(y))W_2\left(\frac{\xi_1+\xi_2}{2}+\nabla\psi_n(y)\right)\Bigg] dy. \end{aligned}$$

Then using the uniform convexity (2.4) of W_1 and W_2, we deduce that

$$\begin{aligned} W_0\left(x,\frac{\xi_1+\xi_2}{2}\right) &\le \frac{1}{2}W_0(x,\xi_1) + \frac{1}{2}W_0(x,\xi_2) \\ &\quad - 2\nu \liminf_{n\to+\infty} \int_Q |\xi_1+\nabla\varphi_n-\xi_2-\nabla\phi_n|^2 \left(1+|\xi_1+\nabla\varphi_n|^2+|\xi_2+\nabla\phi_n|^2\right)^{(p-2)/2} dy. \end{aligned}$$

We now distinguish two cases. First if $p \geq 2$, then by the convexity of the map

$$\mathbb{R}^N \times \mathbb{R}^N \ni (\xi_1, \xi_2) \mapsto |\xi_1 - \xi_2|^2 (1 + |\xi_1|^2 + |\xi_2|^2)^{(p-2)/2},$$

and by lower semicontinuity we deduce that

$$W_0\left(x, \frac{\xi_1 + \xi_2}{2}\right) \leq \frac{1}{2} W_0(x, \xi_1) + \frac{1}{2} W_0(x, \xi_2) - 2\nu |\xi_1 - \xi_2|^2 \left(1 + |\xi_1|^2 + |\xi_2|^2\right)^{(p-2)/2}.$$

If on the other hand $1 < p < 2$, then $p/2 < 1$ and by the reverse Hölder's inequality, we have

$$\begin{aligned} &\int_Q |\xi_1 + \nabla\varphi_n - \xi_2 - \nabla\phi_n|^2 \left(1 + \frac{1}{2}|\xi_1 + \nabla\varphi_n - \xi_2 - \nabla\phi_n|^2\right)^{(p-2)/2} dy \\ &\quad \geq \left(\int_Q |\xi_1 + \nabla\varphi_n - \xi_2 - \nabla\phi_n|^p \, dy\right)^{2/p} \left(\int_Q \left(1 + |\xi_1 + \nabla\varphi_n|^2 + |\xi_2 + \nabla\phi_n|^2\right)^{p/2} dy\right)^{(p-2)/p}. \end{aligned}$$

Then by lower semicontinuity, we have that

$$\liminf_{n\to+\infty} \left(\int_Q |\xi_1 + \nabla\varphi_n - \xi_2 - \nabla\phi_n|^p \, dy\right)^{2/p} \geq |\xi_1 - \xi_2|^2,$$

and according to (5.4), together with the fact that $(p-2)/2 < 0$, we infer that

$$\liminf_{n\to+\infty} \left(\int_Q \left(1 + |\xi_1 + \nabla\varphi_n|^2 + |\xi_2 + \nabla\phi_n|^2\right)^{p/2} dy\right)^{(p-2)/p} \geq c\big(1 + |\xi_1|^2 + |\xi_2|^2\big)^{(p-2)/2},$$

for some constant $c > 0$ depending only on α, β and p. Hence,

$$W_0\left(x, \frac{\xi_1 + \xi_2}{2}\right) \leq \frac{1}{2} W_0(x, \xi_1) + \frac{1}{2} W_0(x, \xi_2) - c|\xi_1 - \xi_2|^2 \big(1 + |\xi_1|^2 + |\xi_2|^2\big)^{(p-2)/2},$$

which completes the proof of the lemma. □

We now define the total energy at the initial time by

$$\mathcal{E}_0 := \int_\Omega Q\psi_0(\nabla u_0)\, dx = \int_\Omega W_0(x, \nabla u_0)\, dx + \kappa \int_\Omega (1 - \Theta_0)\, dx = I_0,$$

where $\Theta_0 := 1 - \theta_0$ is the local volume fraction of the undamaged material.

5.2.2. *Subsequent time steps.* Let $i \geq 1$, and assume that there exist a Carathéodory function $W^k_{i-1} \in \mathcal{F}(\Omega, \alpha, \beta, p)$ and $\Theta^k_{i-1} \in L^\infty(\Omega; [0,1])$ such that $W^k_{i-1}(x, \cdot) \in G_{1-\Theta^k_{i-1}(x)}(W_1, W_2)$ for a.e. $x \in \Omega$. Following [27], at time t^k_i and at every points $x \in \Omega$ of the material, we look for all possible rearrangements between the damaged material W_1 and the one obtained at the previous time step $W^k_{i-1}(x, \cdot)$. The latter has a volume fraction $\Theta^k_{i-1}(x)$ corresponding to the undamaged material W_2, and thus the quantity of dissipated energy paid up to time t^k_{i-1} is $1 - \Theta^k_{i-1}(x)$. Consequently, if the material remains in the same state, the cost of dissipated energy in 0, while if the material becomes damaged, the cost is $\Theta^k_{i-1}(x)$. By irreversibility, there is no other choice, and thus, at time t^k_i one wants to minimize

$$(v, \chi) \mapsto \int_\Omega \Big[\chi W_1(\nabla v) + (1 - \chi) W^k_{i-1}(x, \nabla v) + \kappa\chi\Big]\, dx + \kappa \int_\Omega \Theta^k_{i-1} \chi \, dx,$$

among all $(v, \chi) \in \mathcal{A}(t^k_i) \times L^\infty(\Omega; \{0, 1\})$. We first minimize with respect to χ which leads to the following nonconvex integrand

$$\psi^k_i(x, \xi) := \min\{W_1(\xi) + \kappa\Theta^k_{i-1}(x), W^k_{i-1}(x, \xi)\},$$

and the previous minimization problem is equivalent to

$$I^k_i := \inf_{v \in \mathcal{A}(t^k_i)} \int_\Omega \psi^k_i(x, \nabla v)\, dx. \tag{5.5}$$

Once again, the integrand ψ_i^k is not convex, and thus (5.5) may fail to have solutions. We need then to consider the relaxed problem, and as before it suffices to replace ψ_i^k by its (quasi)convexification defined by

$$Q\psi_i^k(x,\xi) := \inf_{\varphi \in W^{1,p}_{\rm per}(Q)} \int_Q \psi_i^k(x, \xi + \nabla\varphi(y))\, dy,$$

or by Lemma 5.1,

$$Q\psi_i^k(x,\xi) = \min_{\theta\in[0,1]} \left[\min_{W^* \in G_\theta(W_1, W_{i-1}^k(x,\cdot))} W^*(\xi) + \kappa \Theta_{i-1}^k(x)\theta \right], \tag{5.6}$$

with

$$I_i^k = \min_{v \in \mathcal{A}(t_i^k)} \int_\Omega Q\psi_i^k(x, \nabla v)\, dx. \tag{5.7}$$

Let $u_i^k \in \mathcal{A}(t_i^k)$ be a solution of (5.7), then

$$I_i^k = \int_\Omega Q\psi_i^k(x, \nabla u_i^k)\, dx,$$

and there exists $\theta_i^k(x) \in [0,1]$ and $W_i^k(x,\cdot) \in G_{\theta_i^k(x)}(W_1, W_{i-1}^k(x,\cdot))$ such that

$$Q\psi_i^k(x, \nabla u_i^k(x)) = W_i^k(x, \nabla u_i^k(x)) + \kappa\Theta_{i-1}^k(x)\theta_i^k(x)$$

for a.e. $x \in \Omega$. Let us denote by

$$\Theta_i^k := \Theta_{i-1}^k(1-\theta_i^k) \tag{5.8}$$

the volume fraction of the strong (undamaged) material.

Lemma 5.3. *The functions θ_i^k and W_i^k can be chosen to be respectively measurable and Carathéodory. Moreover, $\theta_i^k \in L^\infty(\Omega;[0,1])$, $W_i^k \in \mathcal{F}(\Omega,\alpha,\beta,p)$ and $W_i^k(x,\cdot) \in G_{1-\Theta_i^k(x)}(W_1,W_2)$ is uniformly convex and of class $\mathcal{C}^1$ for a.e. $x \in \Omega$.*

Proof. We first show that

$$W_i^k(x,\cdot) \in G_{1-\Theta_i^k(x)}(W_1,W_2) \quad \text{ for a.e. } x \in \Omega. \tag{5.9}$$

Indeed, since

$$W_i^k(x,\cdot) \in G_{\theta_i^k(x)}(W_1, W_{i-1}^k(x,\cdot)) \text{ and } W_{i-1}^k(x,\cdot) \in G_{1-\Theta_{i-1}^k(x)}(W_1,W_2),$$

then one can find sequences of characteristic functions (χ_i^n) and (χ_{i-1}^n) in $L^\infty(Q;\{0,1\})$ satisfying $\chi_i^n \xrightharpoonup{*} \theta_i^k(x)$ and $\chi_{i-1}^n \xrightharpoonup{*} 1-\Theta_{i-1}^k(x)$ in $L^\infty(Q;[0,1])$, and such that

$$\Gamma\text{-}\lim_{n\to+\infty} \int_\Omega \Big[\chi_i^n(y)W_1(\nabla v(y)) + (1-\chi_i^n(y))W_{i-1}^k(x,\nabla v(y))\Big]\, dy = \int_\Omega W_i^k(x,\nabla v(y))\, dy,$$

and

$$\Gamma\text{-}\lim_{n\to+\infty} \int_\Omega \Big[\chi_{i-1}^n(y)W_1(\nabla v(y)) + (1-\chi_{i-1}^n(y))W_2(\nabla v(y))\Big]\, dy = \int_\Omega W_{i-1}^k(x,\nabla v(y))\, dy.$$

As a consequence of Lemma 4.1, we derive that

$$\begin{aligned}\Gamma\text{-}\lim_{n\to+\infty} \Gamma\text{-}\lim_{m\to+\infty} \int_\Omega \Big[& \big(\chi_i^n(y) + (1-\chi_i^n(y))\chi_{i-1}^m(y)\big)W_1(\nabla v(y) \\ & +(1-\chi_i^n(y))(1-\chi_{i-1}^m(y))W_2(\nabla v(y))\Big]\, dy = \int_\Omega W_i^k(x,\nabla v(y))\, dy,\end{aligned}$$

and for every $\varphi \in L^1(\Omega)$,

$$\lim_{n\to+\infty}\lim_{m\to+\infty} \int_\Omega \big[\chi_i^n(y) + (1-\chi_i^n(y))\chi_{i-1}^m(y)\big]\varphi(y)\, dy = \big(\theta_i^k(x) + (1-\theta_i^k(x))(1-\Theta_{i-1}^k(x))\big)\int_\Omega \varphi(y)\, dy.$$

Using a diagonalization argument together with the metrizability of both $L^\infty(\Omega;[0,1])$ and $\mathcal{S}_\Psi$, we obtain the existence of a sequence $m(n) \to +\infty$ as $n \to +\infty$ such that, setting $\tilde\chi_n := \chi_i^n + (1-\chi_i^n)\chi_{i-1}^{m(n)}$, then $\tilde\chi_n \xrightarrow{*} \theta_i^k(x) + (1-\theta_i^k(x))(1-\Theta_{i-1}^k(x))$ in $L^\infty(\Omega;[0,1])$ and

$$\Gamma\text{-}\lim_{n\to+\infty} \int_\Omega \Big[\tilde\chi^n(y) W_1(\nabla v(y)) + (1-\tilde\chi^n(y)) W_2(\nabla v(y))\Big]\, dy = \int_\Omega W_i^k(x, \nabla v(y))\, dy.$$

Consequently, $W_i^k(x,\cdot) \in \mathcal{G}_{\theta_i^k(x)+(1-\theta_i^k(x))(1-\Theta_{i-1}^k(x))}(W_1,W_2)$, and by the localization property (4.3) of the G-closure set, we conclude that $W_i^k(x,\cdot) \in G_{\theta_i^k(x)+(1-\theta_i^k(x))(1-\Theta_{i-1}^k(x))}(W_1,W_2)$. Then by definition (5.8) of Θ_i^k we deduce that $W_i^k(x,\cdot) \in G_{1-\Theta_i^k(x)}(W_1,W_2)$.

Now thanks to (5.9) and Lemma 4.2, it follows that the function $W_i^k(x,\cdot)$ is of class $\mathcal{C}^1$ and convex. Then a similar argument than that used in the proof of Lemma 5.2 implies that $W_i^k(x,\cdot)$ is actually uniformly convex, and the functions θ_i^k and $W_i^k(\cdot,\xi)$ can be chosen to be measurable. □

We deduce from Lemma 5.3 that

$$I_i^k = \int_\Omega W_i^k(x,\nabla u_i^k)\, dx + \kappa \int_\Omega (\Theta_{i-1}^k - \Theta_i^k)\, dx, \tag{5.10}$$

and we define the total energy at time t_i^k by

$$\mathcal{E}_i^k := I_i^k + \kappa \int_\Omega (1-\Theta_{i-1}^k)\, dx = \int_\Omega W_i^k(x,\nabla u_i^k)\, dx + \kappa \int_\Omega (1-\Theta_i^k)\, dx. \tag{5.11}$$

5.3. **A few properties of the discrete evolution.** We now establish some minimality, monotonicity and energy inequality properties of the discrete evolution, that will used to pass to the limit as the time step tends to zero.

5.3.1. *Minimality.* Since u_i^k is a solution of (5.7) and $W_i^k(x,\cdot) \in G_{\theta_i^k(x)}(W_1, W_{i-1}^k(x,\cdot))$, it follows immediately that u_i^k is also a solution of

$$\inf_{v\in\mathcal{A}(t_i^k)} \int_\Omega W_i^k(x,\nabla v)\, dx. \tag{5.12}$$

5.3.2. *Monotonicity.* We now show that the stored energy of the damaged material decreases as the time increases. This is in agreement with the fact that the irreversible damage process decreases the rigidity of the body.

Lemma 5.4. *For each $i \geq 1$, one has*

$$W_i^k(x,\xi) \leq W_{i-1}^k(x,\xi) \tag{5.13}$$

for all $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

Proof. We first establish that $W_1(\xi) \leq W_i^k(x,\xi)$. From Lemma 5.3 we know that $W_i^k(x,\cdot) \in G_{1-\Theta_i^k(x)}(W_1,W_2)$, and thus there exists a sequence $(\chi_n) \subset L^\infty(Q;\{0,1\})$ with $\int_Q \chi_n(y)\, dy = 1 - \Theta_i^k(x)$ for every $n \in \mathbb{N}$, and

$$\begin{aligned} W_i^k(x,\xi) &= \lim_{n\to+\infty} \inf_{\varphi\in W^{1,p}_{\rm per}(Q)} \int_Q \Big[\chi_n(y) W_1(\xi+\nabla\varphi(y)) + (1-\chi_n(y)) W_2(\xi+\nabla\varphi(y))\Big] dy \\ &\geq \inf_{\varphi\in W^{1,p}_{\rm per}(Q)} \int_Q W_1(\xi+\nabla\varphi(y))\, dy \geq W_1(\xi), \end{aligned}$$

where we used the fact that $W_2 \geq W_1$ in the first inequality, and the convexity of W_1 in the second one.

We are now in position to establish the claimed monotonicity property. Indeed, since $W_i^k(x,\cdot) \in G_{\theta_i^k(x)}(W_1, W_{i-1}^k(x,\cdot))$, then one can find a sequence $(\chi_n) \subset L^\infty(Q;\{0,1\})$ with $\int_Q \chi_n(y)\, dy = \theta_i^k(x)$ for every $n \in \mathbb{N}$, and

$$\begin{aligned} W_i^k(x,\xi) &= \lim_{n\to+\infty} \inf_{\varphi\in W^{1,p}_{\rm per}(Q)} \int_Q \Big[\chi_n(y) W_1(\xi+\nabla\varphi(y)) + (1-\chi_n(y)) W_{i-1}^k(x,\xi+\nabla\varphi(y))\Big] dy \\ &\leq \int_Q (\chi_n(y) W_1(\xi) + (1-\chi_n(y)) W_{i-1}^k(x,\xi))\, dy \leq W_{i-1}^k(x,\xi), \end{aligned}$$

where we used the fact that $W_1(\xi) \leq W^k_{i-1}(x,\xi)$ in the last inequality. $\square$

The next result will be of use to derive the lower bound on the total energy in Proposition 5.3. It is based on a diagonalization argument similar to that of Lemma 5.3.

Lemma 5.5. *For every $j \geq i$, one has*

$$W^k_j(x,\cdot) \in G_{1-\frac{\Theta^k_j(x)}{\Theta^k_i(x)}}(W_1, W^k_i(x,\cdot))$$

for a.e. $x \in \Omega$.

Proof. For every $i \geq 2$, one has

$$W^k_i(x,\cdot) \in G_{\theta^k_i(x)}(W_1, W^k_{i-1}(x,\cdot)) \text{ and } W^k_{i-1}(x,\cdot) \in G_{\theta^k_{i-1}(x)}(W_1, W^k_{i-2}(x,\cdot)).$$

Using a diagonalization argument exactly as in the proof of Lemma 5.3, we can prove that

$$W^k_i(x,\cdot) \in G_{1-\frac{\Theta^k_i(x)}{\Theta^k_{i-2}(x)}}(W_1, W^k_{i-2}(x,\cdot)),$$

and the conclusion follows from an induction argument. $\square$

5.3.3. *Upper bound on the total energy.* We now derive an energy inequality for discrete times which will be used in passing to the time-continuous limit. For every $i \geq 0$, since $W^k_i(x,\cdot) \in G_0(W_1, W^k_i(x,\cdot))$, by definition (5.6) of $Q\psi^k_{i+1}$, we derive that $Q\psi^k_{i+1}(x,\xi) \leq W^k_i(x,\xi)$ for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^N$. Consequently, by (5.7), we deduce that

$$I^k_{i+1} \leq \int_\Omega W^k_i(x, \nabla u^k_i + \nabla g^k_{i+1} - \nabla g^k_i)\, dx.$$

Using the fact that the map

$$t \mapsto \int_\Omega W^k_i(x, \nabla u^k_i + t(\nabla g^k_{i+1} - \nabla g^k_i))\, dx$$

is of class $\mathcal{C}^1$ from $\mathbb{R}$ to $\mathbb{R}$ with derivative given by

$$t \mapsto \int_\Omega DW^k_i(x, \nabla u^k_i + t(\nabla g^k_{i+1} - \nabla g^k_i)) \cdot (\nabla g^k_{i+1} - \nabla g^k_i)\, dx,$$

then by the Mean Value Theorem one can find $s^k_i \in [0,1]$ such that

$$\begin{aligned}\int_\Omega W^k_i(x, \nabla u^k_i + \nabla g^k_{i+1} - \nabla g^k_i)\, dx &= \int_\Omega W^k_i(x, \nabla u^k_i)\, dx \\ &\quad + \int_\Omega DW^k_i(x, \nabla u^k_i + s^k_i(\nabla g^k_{i+1} - \nabla g^k_i)) \cdot (\nabla g^k_{i+1} - \nabla g^k_i)\, dx.\end{aligned}$$

Consequently,

$$I^k_{i+1} \leq I^k_i - \kappa \int_\Omega (\Theta^k_{i-1} - \Theta^k_i)\, dx + \int_\Omega DW^k_i(x, \nabla u^k_i + s^k_i(\nabla g^k_{i+1} - \nabla g^k_i)) \cdot (\nabla g^k_{i+1} - \nabla g^k_i)\, dx,$$

and from the definition (5.11) of the total energy, we get that

$$\mathcal{E}^k_{i+1} \leq \mathcal{E}^k_i + \int_\Omega DW^k_i(x, \nabla u^k_i + s^k_i(\nabla g^k_{i+1} - \nabla g^k_i)) \cdot (\nabla g^k_{i+1} - \nabla g^k_i)\, dx.$$

Hence summing up for $i = 0$ to $j-1$ leads to

$$\mathcal{E}^k_j \leq \mathcal{E}_0 + \sum_{i=0}^{j-1} \int_\Omega DW^k_i(x, \nabla u^k_i + s^k_i(\nabla g^k_{i+1} - \nabla g^k_i)) \cdot (\nabla g^k_{i+1} - \nabla g^k_i)\, dx. \tag{5.14}$$

5.4. **Piecewise constant interpolation.** Before taking the limit as the time step tends to zero, we need to define piecewise constant interpolations for all the discrete time functions considered before. For each $t \in [t_i^k, t_{i+1}^k)$, we set

$$g_k(t) := g_i^k, \quad u_k(t) := u_i^k, \quad \Theta_k(t) := \Theta_i^k, \quad W_k(t) := W_i^k,$$

and we define $\mathcal{A}_k(t) := \mathcal{A}(t_i^k)$ and $\mathcal{E}_k(t) := \mathcal{E}_i^k$. In particular, (5.12) implies that $u_k(t) \in \mathcal{A}_k(t)$ is a solution of

$$\min_{v \in \mathcal{A}_k(t)} \int_\Omega W_k(t)(x, \nabla v)\, dx,$$

the monotonicity property (5.13) and the definition (5.8) of Θ_i^k yield $W_k(t) \in \mathcal{F}(\Omega, \alpha, \beta, p)$, $\Theta_k(t) \in L^\infty(\Omega; [0,1])$ for every $k \in \mathbb{N}$ and $t \in [0,T]$, and

$$W_k(t)(x,\xi) \le W_k(s)(x,\xi) \quad \text{and} \quad \Theta_k(t)(x) \le \Theta_k(s)(x)$$

for every $0 \le s \le t \le T$, every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

Let us define the function $\Phi_k : [0,T] \to L^p(\Omega; \mathbb{R}^N)$ by

$$\Phi_k(t) := s_i^k(\nabla g_{i+1}^k - \nabla g_i^k) = s_i^k \int_{t_i^k}^{t_{i+1}^k} \nabla \dot g(\tau)\, d\tau$$

for $t \in [t_i^k, t_{i+1}^k)$. Then Φ_k is Bochner integrable, and from (5.1) together with the fact that $\nabla \dot g \in L^1(0,T; L^p(\Omega;\mathbb{R}^N))$, we deduce that $\|\Phi_k(t)\|_{L^p(\Omega;\mathbb{R}^N)} \to 0$ uniformly with respect to $t \in [0,T]$. From the energy estimate (5.14), we infer that

$$\mathcal{E}_k(t) \le \mathcal{E}_0 + \int_0^t \int_\Omega DW_k(\tau)(x, \nabla u_k(\tau) + \Phi_k(\tau)) \cdot \nabla \dot g(\tau)\, dx\, d\tau. \tag{5.15}$$

5.5. **The time continuous limit.** We now pass to the limit as the time step tends to zero. We first establish a compactness result on $(u_k(t), \Theta_k(t), W_k(t))$.

Proposition 5.1. *There exist a subsequence (not relabeled) and functions $u(t) \in \mathcal{A}(t)$, $\Theta(t) \in L^\infty(\Omega;[0,1])$ and $W(t) \in \mathcal{F}(\Omega,\alpha,\beta,p)$ such that for every $t \in [0,T]$,*

$$\begin{cases} u_k(t) \rightharpoonup u(t) \text{ in } W^{1,p}(\Omega), \\ \Theta_k(t) \overset{*}{\rightharpoonup} \Theta(t) \text{ in } L^\infty(\Omega;[0,1]), \\ \displaystyle\int_\Omega W(t)(x,\nabla v)\, dx = \Gamma\text{-}\lim_{k\to+\infty} \int_\Omega W_k(t)(x,\nabla v)\, dx. \end{cases}$$

Moreover the map $u : [0,T] \to W^{1,p}(\Omega)$ is strongly measurable and $u \in L^\infty([0,T]; W^{1,p}(\Omega))$, the maps $t \mapsto \Theta(t)$ and $t \mapsto W(t)$ are decreasing, and $W(t)(x,\cdot) \in G_{1-\Theta(t)(x)}(W_1,W_2)$ is uniformly convex and of class $\mathcal{C}^1$ for a.e. $x \in \Omega$ and every $t \in [0,T]$.

Proof. According to [27, Remark 4] and Theorem 3.3 (see also Theorem 3.2 with $L^\infty(\Omega;[0,1])$ endowed with the weak* topology which is a compact metric space), there exist a subsequence (still denoted k) independent of t, $\Theta(t) \in L^\infty(\Omega;[0,1])$ and $W(t) \in \mathcal{F}(\Omega,\alpha,\beta,p)$ such that for every $t \in [0,T]$

$$\begin{cases} \Theta_k(t) \overset{*}{\rightharpoonup} \Theta(t) \quad \text{in } L^\infty(\Omega;[0,1]), \\ \displaystyle\int_\Omega W(t)(x,\nabla v)\, dx = \Gamma\text{-}\lim_{k\to+\infty} \int_\Omega W_k(t)(x,\nabla v)\, dx, \end{cases} \tag{5.16}$$

where $\Theta(t)$ and $W(t)$ are still decreasing with respect to t. Since $W_k(t)(x,\cdot) \in G_{1-\Theta_k(t)(x)}(W_1,W_2)$, by the local character of G-closure (4.3) we deduce that $W_k(t) \in \mathcal{G}_{1-\Theta_k(t)}(W_1,W_2)$, and using a diagonalization argument together with (5.16) and the metrizability of Γ-convergence and $L^\infty(\Omega;[0,1])$ (endowed with the weak* convergence), we infer that $W(t) \in \mathcal{G}_{1-\Theta(t)}(W_1,W_2)$. Hence, using again (4.3) we obtain that $W(t)(x,\cdot) \in G_{1-\Theta(t)(x)}(W_1,W_2)$. Moreover, as in the proof of Lemma 5.2, one can show that $W(t)(x,\cdot)$ is uniformly convex.

From the minimality property satisfied by $u_k(t)$, the fact that the sequence (g_k) is bounded in $L^\infty(0,T;W^{1,p}(\Omega))$ and Poincaré's inequality, we infer that

$$\sup_{t\in[0,T]}\sup_{k\in\mathbb{N}}\|u_k(t)\|_{W^{1,p}(\Omega)}<+\infty. \tag{5.17}$$

For each $t\in[0,T]$, there exist a t-dependent subsequence (k_j) and $u(t)\in W^{1,p}(\Omega)$ such that

$$u_{k_j}(t)\rightharpoonup u(t) \quad \text{in } W^{1,p}(\Omega).$$

Moreover, the continuity of the trace and the fact that $g_k(t)\to g(t)$ strongly in $W^{1,p}(\Omega)$ ensure that $u(t)=g(t)$ $\mathcal{H}^{N-1}$-a.e. on $\partial_D\Omega$, so that $u(t)\in\mathcal{A}(t)$. From standard properties of Γ-convergence we get that $u(t)$ is a solution of

$$\min_{v\in\mathcal{A}(t)}\int_\Omega W(t)(x,\nabla v)\,dx. \tag{5.18}$$

Note that since $W(t)(x,\cdot)$ is in particular strictly convex, then the previous minimization problem admits a unique solution which must be $u(t)$. Hence by uniqueness the whole sequence $u_k(t)$ weakly converges to $u(t)$ in $W^{1,p}(\Omega)$. If $L\in[W^{1,p}(\Omega)]'$, then $t\mapsto\langle L,u_k(t)\rangle$ is a simple scalar valued measurable function, and since $u_k(t)\rightharpoonup u(t)$ in $W^{1,p}(\Omega)$ for each $t\in[0,T]$, then $\langle L,u_k(t)\rangle\to\langle T,u(t)\rangle$ for every $t\in[0,T]$ which proves that the map $t\mapsto\langle L,u(t)\rangle$ is measurable. Consequently, $t\mapsto u(t)$ is weakly measurable from $[0,T]$ to $W^{1,p}(\Omega)$, and according to the Pettis Theorem (see [25, Theorem 2.104]) together with the separability of $W^{1,p}(\Omega)$, we deduce that $t\mapsto u(t)$ is actually strongly measurable from $[0,T]$ to $W^{1,p}(\Omega)$. Thanks to (5.17), we get that

$$\sup_{t\in[0,T]}\|u(t)\|_{W^{1,p}(\Omega)}<+\infty,$$

and thus $u\in L^\infty([0,T];W^{1,p}(\Omega))$. □

We define the total energy at time $t\in[0,T]$ by

$$\mathcal{E}(t):=\int_\Omega W(t)(x,\nabla u(t))\,dx+\kappa\int_\Omega(1-\Theta(t))\,dx.$$

Our next goal is to prove the energy balance. We first prove a technical lemma which will allow us to apply Lemmas 3.1 and 3.2 to get a first energy inequality. Note that this is the only part of the proof where the uniform convexity is really essential. Indeed, this hypothesis ensures the strong continuity of the map $\mathbb{R}^N\ni\xi\mapsto\varphi_\xi\in W^{1,p}_{\rm per}(Q)$, where φ_ξ is the (unique) solution of the cell problem (4.2).

Lemma 5.6. *The following properties hold:*

(i) *There exists a constant $\gamma>0$ such that for every $k\in\mathbb{N}$, $\xi\in\mathbb{R}^N$, $t\in[0,T]$ and a.e. $x\in\Omega$, then*

$$|DW_k(t)(x,\xi)|\le\gamma(1+|\xi|^{p-1});$$

(ii) *For any sequences (ξ_k) and $(\xi'_k)\subset\mathbb{R}^N$ such that $|\xi_k|\le M$, $|\xi'_k|\le M$ and $|\xi_k-\xi'_k|\to0$ as $k\to+\infty$, for some $M>0$, then*

$$|DW_k(t)(x,\xi_k)-DW_k(t)(x,\xi'_k)|\to0,$$

for every $t\in[0,T]$ and a.e. $x\in\Omega$.

Proof. The first item is a consequence of the fact that $W_k(t)(x,\cdot)\in\mathcal{F}(\alpha,\beta,p)$ is a convex function of class $\mathcal{C}^1$ together with (2.3).

We now investigate the proof of (ii). By a diagonalization argument, there is no loss of generality to assume that $W_k(t)(x,\cdot)\in P_{1-\Theta_k(t)(x)}(W_1,W_2)$. From Lemmas 5.2 and 5.3 we know that for each $k\in\mathbb{N}$, $t\in[0,T]$ and a.e. $x\in\Omega$, the function $W_k(t)(x,\cdot)$ is uniformly convex and of class $\mathcal{C}^1$. Thus by (2.5), there exists $\nu'>0$ such that

$$(DW_k(t)(x,\xi_2)-DW_k(t)(x,\xi_1))\cdot(\xi_2-\xi_1)\ge\nu'|\xi_1-\xi_2|^2(1+|\xi_1|^2+|\xi_2|^2)^{(p-2)/2} \tag{5.19}$$

for every ξ_1 and $\xi_2 \in \mathbb{R}^N$. Let $M > 0$ and ξ_k, $\xi_k' \in \mathbb{R}^N$ such that $|\xi_k| \le M$, $|\xi_k'| \le M$ and $|\xi_k - \xi_k'| \to 0$ as $k \to +\infty$. Consider $\chi_k \in L^\infty(Q;\{0,1\})$ such that $\int_Q \chi_k(y)\,dy = 1 - \Theta_k(t)(x)$ and $W_k(t)(x,\cdot) = (W_{\chi_k})_{\rm hom}$. Let φ_{ξ_k} and $\varphi_{\xi_k'} \in W^{1,p}_{\rm per}(Q)$ satisfying

$$W_k(t)(x,\xi_k) = \int_Q W_{\chi_k}(y,\xi_k + \nabla\varphi_{\xi_k})\,dy \quad \text{and} \quad W_k(t)(x,\xi_k') = \int_Q W_{\chi_k}(y,\xi_k' + \nabla\varphi_{\xi_k'})\,dy.$$

By the p-growth and p-coercivity conditions (2.1), we have that

$$\|\nabla\varphi_{\xi_k}\|_{L^p(Q;\mathbb{R}^N)} \le C(1+|\xi_k|) \text{ and } \|\nabla\varphi_{\xi_k'}\|_{L^p(Q;\mathbb{R}^N)} \le C(1+|\xi_k'|), \tag{5.20}$$

for some constant $C > 0$ independent of k. Note that since W_1 and W_2 are uniformly convex, then $W_{\chi_k}(y,\cdot)$ is in particular strictly convex, and thus the solutions φ_{ξ_k} and $\varphi_{\xi_k'}$ of the cell problem (4.2) are actually unique. Moreover, the weak formulation of the Euler-Lagrange equation gives

$$\int_Q DW_{\chi_k}(y,\xi_k + \nabla\varphi_{\xi_k}) \cdot \nabla\phi\,dy = 0 \quad \text{and} \quad \int_Q DW_{\chi_k}(y,\xi_k' + \nabla\varphi_{\xi_k'}) \cdot \nabla\phi\,dy = 0, \tag{5.21}$$

for every $\phi \in W^{1,p}_{\rm per}(Q)$.

For any $v \in \mathbb{S}^{N-1}$ and any $t \in \mathbb{R}$, one has

$$\frac{W_k(t)(x,\xi_k + tv) - W_k(t)(x,\xi_k)}{t} \le \int_Q \frac{W_{\chi_k}(y,\xi_k + tv + \varphi_{\xi_k}) - W_{\chi_k}(y,\xi_k + \nabla\varphi_{\xi_k})}{t} dy.$$

Since $W_{\chi_k} \in \mathcal{F}(Q,\alpha,\beta,p)$ by the p-growth condition (2.1), (5.20) and Fatou's Lemma, we get letting $t \to 0$,

$$DW_k(t)(x,\xi_k)\cdot v \le \int_Q DW_{\chi_k}(y,\xi_k + \nabla\varphi_{\xi_k})\,dy \cdot v,$$

and thus changing v in $-v$, we infer that

$$DW_k(t)(x,\xi_k)\cdot v = \int_Q DW_{\chi_k}(y,\xi_k + \nabla\varphi_{\xi_k})\,dy \cdot v.$$

Consequently, we proved that

$$DW_k(t)(x,\xi_k) = \int_Q DW_{\chi_k}(y,\xi_k + \nabla\varphi_{\xi_k})\,dy \tag{5.22}$$

and similarly

$$DW_k(t)(x,\xi_k') = \int_Q DW_{\chi_k}(y,\xi_k' + \nabla\varphi_{\xi_k'})\,dy. \tag{5.23}$$

By (5.19) and (5.21), we get that

$$\begin{aligned}
\nu' \int_Q &|(\xi_k - \xi_k') + (\nabla\varphi_{\xi_k} - \nabla\varphi_{\xi_k'})|^2\big(1 + |\xi_k + \nabla\varphi_{\xi_k}|^2 + |\xi_k' + \nabla\varphi_{\xi_k'}|^2\big)^{(p-2)/2}\,dy \\
&\le \int_Q \big(DW_{\chi_k}(y,\xi_k + \nabla\varphi_{\xi_k}) - DW_{\chi_k}(y,\xi_k' + \nabla\varphi_{\xi_k'})\big)\cdot\big((\xi_k + \nabla\varphi_{\xi_k}) - (\xi_k' + \nabla\varphi_{\xi_k'})\big)\,dy \\
&\qquad = \int_Q \big(DW_{\chi_k}(y,\xi_k + \nabla\varphi_{\xi_k}) - DW_{\chi_k}(y,\xi_k' + \nabla\varphi_{\xi_k'})\big)\cdot(\xi_k - \xi_k')\,dy.
\end{aligned}$$

Therefore, since DW_{χ_k} satisfies the $(p-1)$-growth condition (2.3), we deduce that

$$\begin{aligned}
\int_Q &|(\xi_k - \xi_k') + (\nabla\varphi_{\xi_k} - \nabla\varphi_{\xi_k'})|^2\big(1 + |\xi_k + \nabla\varphi_{\xi_k}|^2 + |\xi_k' + \nabla\varphi_{\xi_k'}|^2\big)^{(p-2)/2}\,dy \\
&\le C_1(1 + |\xi_k|^{p-1} + |\xi_k'|^{p-1})|\xi_k - \xi_k'|, \qquad (5.24)
\end{aligned}$$

for some constant $C_1 > 0$ independent of k. We claim that $\nabla\varphi_{\xi_k} - \nabla\varphi_{\xi_k'} \to 0$ in $L^1(Q;\mathbb{R}^N)$. To show this property, we first consider exponents $p \ge 2$. Then from (5.24), since $(p-2)/p \ge 0$, we immediately get that

$$\int_Q |\nabla\varphi_{\xi_k} - \nabla\varphi_{\xi_k'}|^2\,dy \le C_2|\xi_k - \xi_k'|,$$

where $C_2 > 0$ is a constant depending only on p, N, M and ν'. On the other hand, if $1 < p < 2$, then $p/2 < 1$, and by the reverse Hölder's Inequality (see *e.g.* [1, Theorem 2.6]), we have

$$\begin{aligned}
&\int_Q |(\xi_k - \xi'_k) + (\nabla\varphi_{\xi_k} - \nabla\varphi_{\xi'_k})|^2 \big(1 + |\xi_k + \nabla\varphi_{\xi_k}|^2 + |\xi'_k + \nabla\varphi_{\xi'_k}|^2\big)^{(p-2)/2}\, dy \\
&\geq \left(\int_Q |(\xi_k - \xi'_k) + (\nabla\varphi_{\xi_k} - \nabla\varphi_{\xi'_k})|^p dy\right)^{2/p} \left(\int_Q \big(1 + |\xi_k + \nabla\varphi_{\xi_k}|^2 + |\xi'_k + \nabla\varphi_{\xi'_k}|^2\big)^{p/2} dy\right)^{(p-2)/p} \\
&\qquad\qquad \geq C_3(1 + M^p)^{(p-2)/p} \left(\int_Q |(\xi_k - \xi'_k) + (\nabla\varphi_{\xi_k} - \nabla\varphi_{\xi'_k})|^p dy\right)^{2/p},
\end{aligned}$$

for some constant $C_3 > 0$ (independent of k), where we used (5.20) and the fact that $(p-2)/p < 0$. Then using again (5.24), we deduce that

$$\int_Q |\nabla\varphi_{\xi_k} - \nabla\varphi_{\xi'_k}|^p\, dy \leq C_4|\xi_k - \xi'_k|,$$

where $C_4 > 0$ is a constant which only depends on p, N, M and ν'. Gathering both cases, it turns out that indeed $\nabla\varphi_{\xi_k} - \nabla\varphi_{\xi'_k} \to 0$ in $L^1(Q;\mathbb{R}^N)$ and also a.e. (up to a subsequence). From (4.4) (with χ_k instead of χ_n) together with (5.22) and (5.23), we get that

$$\begin{aligned}
|DW_k(t)(x,\xi_k) - DW_k(t)(x,\xi'_k)| &\leq \int_Q |DW_{\chi_k}(y,\xi_k + \nabla\varphi_{\xi_k}) - DW_{\chi_k}(y,\xi'_k + \nabla\varphi_{\xi'_k})|\, dy \\
&\leq \int_Q |DW_1(\xi_k + \nabla\varphi_{\xi_k}) - DW_1(\xi'_k + \nabla\varphi_{\xi'_k})|\, dy \\
&\quad + \int_Q |DW_2(\xi_k + \nabla\varphi_{\xi_k}) - DW_2(\xi'_k + \nabla\varphi_{\xi'_k})|\, dy. \qquad (5.25)
\end{aligned}$$

It remains to show that the right hand side of (5.25) is infinitesimal. To do that, we split the above integrals as a first integral over the set

$$A_R^k := \{x \in Q : |\xi_k + \nabla\varphi_{\xi_k}(x)| \leq R \text{ and } |\xi'_k + \nabla\varphi_{\xi'_k}(x)| \leq R\},$$

where $R > 1$, and a second one over the complementary $Q \setminus A_R^k$, and we estimate separately both terms. Concerning the integral over A_R^k, we denote by $\omega_{1,R}$ (resp. $\omega_{2,R}$) the modulus of continuity of DW_1 (resp. DW_2) in B_R. Since DW_1 and DW_2 are uniformly continuous on B_R, then $\omega_{1,R}(t) \to 0$ and $\omega_{2,R}(t) \to 0$ as $t \to 0$. Then for $i = 1$ and 2, we get from the Dominated Convergence Theorem that

$$\int_{A_R^k} |DW_i(\xi_k + \nabla\varphi_{\xi_k}) - DW_i(\xi'_k + \nabla\varphi_{\xi'_k})|\, dy \leq \int_{A_R^k} \omega_{i,R}(|\xi_k - \xi'_k| + |\nabla\varphi_{\xi_k} - \nabla\varphi_{\xi'_k}|)\, dy \to 0 \quad (5.26)$$

as $k \to 0$ for fixed $R > 1$. We now deal with the integral over $Q \setminus A_R^k$. We first remark that thanks to (5.20), there exists a constant $C > 0$ (independent of k and R) such that $\mathcal{L}^N(Q \setminus A_R^k) \leq C/R^p$. Therefore, using the $(p-1)$-growth condition (2.3) satisfied by W_i, Hölder's inequality and (5.20), we get that

$$\begin{aligned}
&\int_{Q\setminus A_R^k} |DW_i(\xi_k + \nabla\varphi_{\xi_k}) - DW_i(\xi'_k + \nabla\varphi_{\xi'_k})|\, dy \\
&\qquad \leq \gamma \int_{Q\setminus A_R^k} (2 + |\xi_k + \nabla\varphi_{\xi_k}|^{p-1} + |\xi'_k + \nabla\varphi_{\xi'_k}|^{p-1})\, dy \\
&\qquad \leq C\left(1 + \|\xi_k + \nabla\varphi_{\xi_k}\|^{p-1}_{L^p(Q;\mathbb{R}^N)} + \|\xi'_k + \nabla\varphi_{\xi'_k}\|^{p-1}_{L^p(Q;\mathbb{R}^N)}\right)\mathcal{L}^N(Q \setminus A_R^k)^{1/p} \\
&\qquad \leq \frac{C(1 + M^{p-1})}{R} \to 0 \qquad (5.27)
\end{aligned}$$

as $R \to +\infty$, uniformly with respect to k. Now gathering (5.26) and (5.27), and passing to the limit first as $k \to +\infty$ and then as $R \to +\infty$, we obtain that

$$\lim_{k\to+\infty} \int_Q |DW_i(\xi_k + \nabla\varphi_{\xi_k}) - DW_i(\xi'_k + \nabla\varphi_{\xi'_k})|\, dy = 0,$$

which implies, in view of (5.25), that $|DW_k(t)(x,\xi_k) - DW_k(t)(x,\xi'_k)| \to 0$. □

Remark 5.1. As noticed in section 2.2, the functions $W_1(\xi) = \alpha|\xi|^p$ and $W_2(\xi) = \beta|\xi|^p$ (with $p > 1$ and $0 < \alpha \le \beta < +\infty$) do not fulfill the assumption of uniform convexity, although they are very important examples. However, the proof of Lemma 5.6 still works in that case since the differentials DW_1 and DW_2 (and thus also $DW_k(t)(x,\cdot)$) are strongly monotone by (2.6) and (2.7).

We are now in position to state a first energy inequality.

Proposition 5.2. *For any $t \in [0,T]$, one has $\mathcal{E}_k(t) \to \mathcal{E}(t)$, and*

$$\mathcal{E}(t) \le \mathcal{E}_0 + \int_0^t \int_\Omega DW(\tau)(x,\nabla u(\tau)) \cdot \nabla \dot g(\tau)\, dx\, d\tau. \tag{5.28}$$

Proof. By the Γ-convergence result (5.16), we have that

$$\mathcal{E}(t) \le \liminf_{k\to+\infty} \left\{ \int_\Omega W_k(t)(x,\nabla u_k(t))\, dx + \kappa \int_\Omega (1-\Theta_k(t))\, dx \right\},$$

hence from the upper bound estimate (5.15), we get that

$$\mathcal{E}(t) \le \mathcal{E}_0 + \limsup_{k\to+\infty} \int_0^t \int_\Omega DW_k(\tau)(x,\nabla u_k(\tau) + \Phi_k(\tau)) \cdot \nabla \dot g(\tau)\, dx\, d\tau.$$

As a consequence of Lemma 3.1 (with $\mu = \mathcal{L}^{N+1} \llcorner (\Omega \times [0,T])$ and $\mathcal{A} = \mathcal{L}(\Omega) \otimes \mathcal{B}([0,T])$, where $\mathcal{L}(\Omega)$ is the σ-algebra of all Lebesgue measurable subsets of Ω, and $\mathcal{B}([0,T])$ is the σ-algebra of all Borel subsets of $[0,T]$) and Lemma 5.6, since $\Phi_k \to 0$ in $L^p(\Omega \times (0,T);\mathbb{R}^N)$, we infer that

$$\mathcal{E}(t) \le \mathcal{E}_0 + \limsup_{k\to+\infty} \int_0^t \int_\Omega DW_k(\tau)(x,\nabla u_k(\tau)) \cdot \nabla \dot g(\tau)\, dx\, d\tau. \tag{5.29}$$

Next we would like to apply Lemma 3.2 to deduce (5.28) from (5.29). To do that, we need to show the convergence of the elastic energy. First, by the Γ-lower bound, we have

$$\int_\Omega W(t)(x,\nabla u(t))\, dx \le \liminf_{k\to+\infty} \int_\Omega W_k(t)(x,\nabla u_k(t))\, dx. \tag{5.30}$$

We now prove that this inequality is actually an equality. Consider a recovery sequence $(u_k^*) \subset W^{1,p}(\Omega)$ such that $u_k^* \rightharpoonup u(t)$ in $W^{1,p}(\Omega)$ and

$$\int_\Omega W(t)(x,\nabla u(t))\, dx = \lim_{k\to+\infty} \int_\Omega W_k(t)(x,\nabla u_k^*)\, dx.$$

Since $g_k(t) \to g(t)$ strongly in $W^{1,p}(\Omega)$ and $u(t) = g(t)$ on $\partial_D\Omega$, by a standard truncation argument which uses a suitable cut-off function, there is no loss of generality to assume that $u_k^* = g_k(t)$ on $\partial_D\Omega$ so that $u_k^*(t) \in \mathcal{A}_k(t)$. Hence by minimality of $u_k(t)$ we obtain that for each $k \in \mathbb{N}$,

$$\int_\Omega W_k(t)(x,\nabla u_k(t))\, dx \le \int_\Omega W_k(t)(x,\nabla u_k^*)\, dx,$$

and thus, taking the limsup as $k \to +\infty$, we deduce that

$$\limsup_{k\to+\infty} \int_\Omega W_k(t)(x,\nabla u_k(t))\, dx \le \int_\Omega W(t)(x,\nabla u(t))\, dx. \tag{5.31}$$

Hence gathering (5.30) and (5.31), we get that

$$\lim_{k\to+\infty} \int_\Omega W_k(t)(x,\nabla u_k(t))\, dx = \int_\Omega W(t)(x,\nabla u(t))\, dx,$$

and consequently $\mathcal{E}_k(t) \to \mathcal{E}(t)$. We are now in position to apply Lemma 3.2 from which we obtain that for every $t \in [0,T]$,

$$\lim_{k\to+\infty} \int_\Omega DW_k(t)(x,\nabla u_k(t)) \cdot \nabla \dot g(t)\, dx = \int_\Omega DW(t)(x,\nabla u(t)) \cdot \nabla \dot g(t)\, dx,$$

and using (5.29) together with the Dominated Convergence Theorem, we get the desired energy inequality (5.28). □

We next prove that the energy inequality (5.28) is actually an equality. Following [17] the argument rests on the approximation of the Bochner integral by suitable Riemann sums (see [17, Lemma 4.12]) that we recall here.

Lemma 5.7. *Let X be a Banach space and $f : [0,T] \to X$ be a Bochner integrable function. Then there exists a sequence of subdivisions $(s_i^k)_{0\le i\le i_k}$ of the interval $[0,T]$, with*

$$0 = s_0^k < s_1^k < \ldots < s_{i_k-1}^k < s_{i_k}^k = T \quad \text{and} \quad \lim_{k\to+\infty} \max_{1\le i\le i_k} (s_i^k - s_{i-1}^k) = 0,$$

such that

$$\lim_{k\to+\infty} \sum_{i=1}^{i_k} \int_{s_{i-1}^k}^{s_i^k} \|f(s_i^k) - f(t)\|_X \, dt = 0.$$

Now thanks to Lemma 5.7 we are in position to show the following lower bound inequality on the total energy.

Proposition 5.3. *For any $t \in [0,T]$, then*

$$\mathcal{E}(t) \ge \mathcal{E}_0 + \int_0^t \int_\Omega DW(\tau)(x, \nabla u(\tau)) \cdot \nabla \dot g(\tau) \, dx \, d\tau.$$

Proof. Let $s < t$. Since $u(t) + g(s) - g(t) \in \mathcal{A}(s)$, by Γ-convergence, one can find a sequence (v_k) such that $v_k \in \mathcal{A}_k(s)$ for each $k \in \mathbb{N}$, $v_k \rightharpoonup u(t) + g(s) - g(t)$ in $W^{1,p}(\Omega)$, and

$$\lim_{k\to+\infty} \int_\Omega W_k(t)(x, \nabla v_k) \, dx = \int_\Omega W(t)(x, \nabla u(t) + \nabla g(s) - \nabla g(t)) \, dx. \tag{5.32}$$

Let $i \le j$ be such that $s \in [t_i^k, t_{i+1}^k)$ and $t \in [t_j^k, t_{j+1}^k)$. Since by Lemma 5.5

$$W_j^k(x,\cdot) \in G_{1-\frac{\Theta_j^k(x)}{\Theta_{i-1}^k(x)}} (W_1, W_{i-1}^k(x,\cdot))$$

for a.e. $x \in \Omega$, then from (5.6), (5.7) and (5.10) we have that

$$\int_\Omega W_i^k(x, \nabla u_i^k) \, dx + \kappa \int_\Omega (\Theta_{i-1}^k - \Theta_i^k) \, dx \le \int_\Omega W_j^k(x, \nabla v_k) \, dx + \kappa \int_\Omega \Theta_{i-1}^k \left(1 - \frac{\Theta_j^k(x)}{\Theta_{i-1}^k(x)}\right) dx,$$

and thus

$$\int_\Omega W_k(s)(x, \nabla u_k(s)) \, dx + \kappa \int_\Omega (1 - \Theta_k(s)) dx \le \int_\Omega W_k(t)(x, \nabla v_k) \, dx + \kappa \int_\Omega (1 - \Theta_k(t)) dx.$$

Hence taking the limit as $k \to +\infty$, (5.32) leads to

$$\int_\Omega W(s)(x, \nabla u(s)) \, dx + \kappa \int_\Omega (1 - \Theta(s)) dx \\ \le \int_\Omega W(t)(x, \nabla u(t) + \nabla g(s) - \nabla g(t)) \, dx + \kappa \int_\Omega (1 - \Theta(t)) dx,$$

and consequently,

$$\mathcal{E}(s) \le \mathcal{E}(t) + \int_\Omega W(t)(x, \nabla u(t) + \nabla g(s) - \nabla g(t)) \, dx - \int_\Omega W(t)(x, \nabla u(t)) \, dx.$$

By the Mean Value Theorem, there exists $\rho(s,t) \in [0,1]$ such that

$$\mathcal{E}(t) - \mathcal{E}(s) \ge \int_\Omega \left[DW(t) \left(x, \nabla u(t) - \rho(s,t) \int_s^t \nabla \dot g(\tau) \, d\tau \right) \cdot \int_s^t \nabla \dot g(\tau) \, d\tau \right] dx. \tag{5.33}$$

By Lemma 5.7, consider a sequence of subdivisions $(s_i^k)_{0\le i\le i_k}$ of the interval $[0,t]$, with

$$0 = s_0^k < s_1^k < \ldots < s_{i_k-1}^k < s_{i_k}^k = t \quad \text{and} \quad \lim_{k\to+\infty} \max_{1\le i\le i_k} (s_i^k - s_{i-1}^k) = 0,$$

such that

$$\lim_{k\to+\infty} \sum_{i=1}^{i_k} \left\| (s_i^k - s_{i-1}^k) \nabla \dot g(s_i^k) - \int_{s_{i-1}^k}^{s_i^k} \nabla \dot g(t) \, dt \right\|_{L^p(\Omega;\mathbb{R}^N)} = 0 \tag{5.34}$$

and

$$\lim_{k\to+\infty}\sum_{i=1}^{i_k}\left|(s_i^k-s_{i-1}^k)\vartheta(s_i^k)-\int_{s_{i-1}^k}^{s_i^k}\vartheta(t)\,dt\right|=0, \tag{5.35}$$

where

$$\vartheta(t)=\int_\Omega DW(t)(x,\nabla u(t))\cdot\nabla\dot g(t)\,dx. \tag{5.36}$$

For all $s\in[s_i^k,s_{i+1}^k)$, we define

$$\overline{W}_k(s):=W(s_{i+1}^k),\quad \bar u_k(s):=u(s_{i+1}^k)\quad\text{and}\quad \overline{\Psi}_k(s):=-\rho(s_i^k,s_{i+1}^k)\int_{s_i^k}^{s_{i+1}^k}\nabla\dot g(\tau)\,d\tau.$$

As $\nabla\dot g\in L^1(0,T;L^p(\Omega;\mathbb{R}^N))$, we have that

$$\|\overline{\Psi}_k(s)\|_{L^p(\Omega;\mathbb{R}^N)}\to 0$$

uniformly with respect to $s\in[0,t]$. In (5.33) we replace s by s_i^k and t by s_{i+1}^k, then summing up for $i=0$ to i_k-1 yields

$$\mathcal{E}(t)-\mathcal{E}(0)\geq\int_0^t\int_\Omega D\overline{W}_k(\tau)(x,\nabla\bar u_k(\tau)+\overline{\Psi}_k(\tau))\cdot\nabla\dot g(\tau)\,dx\,d\tau.$$

Note that $D\overline{W}_k(t)$ obviously satisfies the $(p-1)$-growth condition (2.3). Moreover, since for $t\in[s_i^k,s_{i+1}^k)$, one has $\overline{W}_k(t)(x,\cdot)=W(s_{i+1}^k)(x,\cdot)\in G_{1-\Theta(s_{i+1}^k)(x)}(W_1,W_2)$ for a.e. $x\in\Omega$, we deduce from similar arguments than those used in the proof of Lemma 5.6 that for all $M>0$ and all sequences (ξ_k) and (ξ'_k) in $\mathbb{R}^N$ satisfying $|\xi_k|\leq M$, $|\xi'_k|\leq M$ and $|\xi_k-\xi'_k|\to 0$, then

$$|D\overline{W}_k(t)(x,\xi_k)-D\overline{W}_k(t)(x,\xi'_k)|\to 0$$

for a.e. $x\in\Omega$. As a consequence, applying Lemma 3.1 and remembering that $\overline{\Psi}_k\to 0$ in $L^p(\Omega\times(0,T);\mathbb{R}^N)$, we get that

$$\mathcal{E}(t)-\mathcal{E}(0)\geq\limsup_{k\to+\infty}\int_0^t\int_\Omega D\overline{W}_k(\tau)(x,\nabla\bar u_k(\tau))\cdot\nabla\dot g(\tau)\,dx\,d\tau. \tag{5.37}$$

By the $(p-1)$-growth condition (2.3) satisfied by $DW(s_{i+1}^k)$ together with Hölder's inequality and (5.34), we have

$$\begin{aligned}\sum_{i=0}^{i_k-1}&\left|\int_\Omega DW(s_{i+1}^k)(x,\nabla u(s_{i+1}^k))\cdot\left[(s_{i+1}^k-s_i^k)\nabla\dot g(s_{i+1}^k)-\int_{s_i^k}^{s_{i+1}^k}\nabla\dot g(\tau)\,d\tau\right]dx\right|\\&\leq C\left(1+\|\nabla u\|^{p-1}_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^N))}\right)\sum_{i=0}^{i_k-1}\left\|(s_{i+1}^k-s_i^k)\nabla\dot g(s_{i+1}^k)-\int_{s_i^k}^{s_{i+1}^k}\nabla\dot g(t)\,dt\right\|_{L^p(\Omega;\mathbb{R}^N)}\to 0,\end{aligned}$$

and thus from (5.35), (5.36) and (5.37) we deduce that

$$\begin{aligned}\mathcal{E}(t)-\mathcal{E}(0)&\geq\limsup_{k\to+\infty}\sum_{i=0}^{i_k-1}(s_{i+1}^k-s_i^k)\int_\Omega DW(s_{i+1}^k)(x,\nabla u(s_{i+1}^k))\cdot\nabla\dot g(s_{i+1}^k)\,dx\\&=\int_0^t\int_\Omega DW(\tau)(x,\nabla u(\tau))\cdot\nabla\dot g(\tau)\,dx\,d\tau,\end{aligned}$$

which completes the proof of the proposition. □

Next, we prove a unilateral minimality property satisfied by the triple $(u(t),\Theta(t),W(t))$ which is stronger than (5.18).

Proposition 5.4. *Let $t\in[0,T]$. For any $v\in\mathcal{A}(t)$, any $\theta\in L^\infty(\Omega;[0,1])$ and any $W\in\mathcal{F}(\Omega,\alpha,\beta,p)$ such that $W(x,\cdot)\in G_{\theta(x)}(W_1,W(t)(x,\cdot))$ for a.e. $x\in\Omega$, then*

$$\int_\Omega W(t)(x,\nabla u(t))\,dx\leq\int_\Omega W(x,\nabla v)\,dx+\kappa\int_\Omega\Theta(t)\theta\,dx.$$

Proof. Let $t \in [0,T]$, $v \in \mathcal{A}(t)$, $\theta \in L^\infty(\Omega;[0,1])$ and $W \in \mathcal{F}(\Omega,\alpha,\beta,p)$ such that $W(x,\cdot) \in G_{\theta(x)}(W_1, W(t)(x,\cdot))$ for a.e. $x \in \Omega$. By Proposition 4.1, there exists a sequence of characteristic functions $(\chi_n) \subset L^\infty(\Omega;\{0,1\})$ such that $\chi_n \xrightharpoonup{*} \theta$ in $L^\infty(\Omega;[0,1])$, and

$$\int_\Omega W(x,\nabla v)\,dx = \Gamma\text{-}\lim_{n\to+\infty}\int_\Omega \Big[\chi_n W_1(\nabla v) + (1-\chi_n)W(t)(x,\nabla v)\Big]\,dx.$$

Moreover, since

$$\int_\Omega W(t)(x,\nabla v)\,dx = \Gamma\text{-}\lim_{k\to+\infty}\int_\Omega W_k(t)(x,\nabla v)\,dx,$$

we deduce from Lemma 4.1 that for each $n \in \mathbb{N}$,

$$\int_\Omega \Big[\chi_n W_1(\nabla v) + (1-\chi_n)W(t)(x,\nabla v)\Big]\,dx = \Gamma\text{-}\lim_{k\to+\infty}\int_\Omega \Big[\chi_n W_1(\nabla v) + (1-\chi_n)W_k(t)(x,\nabla v)\Big]\,dx.$$

Let $(v_n) \subset W^{1,p}(\Omega)$ be a recovery sequence weakly converging to v in $W^{1,p}(\Omega)$ such that

$$\int_\Omega W(x,\nabla v)\,dx = \lim_{n\to+\infty}\int_\Omega \Big[\chi_n W_1(\nabla v_n) + (1-\chi_n)W(t)(x,\nabla v_n)\Big]\,dx. \tag{5.38}$$

Without loss of generality, one can assume that for each $n \in \mathbb{N}$, $v_n = v$ on a neighborhood of $\partial\Omega$, and in particular $v_n \in \mathcal{A}(t)$ for each $n \in \mathbb{N}$. Consider now $(v_{n,k})_k \subset W^{1,p}(\Omega)$ such that $v_{n,k} \rightharpoonup v_n$ in $W^{1,p}(\Omega)$ as $k \to +\infty$, and satisfying

$$\int_\Omega \Big[\chi_n W_1(\nabla v_n) + (1-\chi_n)W(t)(x,\nabla v_n)\Big]\,dx = \lim_{k\to+\infty}\int_\Omega \Big[\chi_n W_1(\nabla v_{n,k}) + (1-\chi_n)W_k(t)(x,\nabla v_{n,k})\Big]\,dx. \tag{5.39}$$

Once again, since for every $t \in [0,T]$, $g_k(t)$ converges strongly to $g(t)$ in $W^{1,p}(\Omega)$ as $k \to +\infty$, it is not restrictive to assume that $v_{n,k} = g_k(t)$ on $\partial_D\Omega$ so that $v_{n,k} \in \mathcal{A}_k(t)$.

Fix $k \in \mathbb{N}$ and let i be such that $t \in [t_i^k, t_{i+1}^k)$, by (5.7) and (5.10), we have that

$$\int_\Omega W_i^k(x,\nabla u_i^k)\,dx + \kappa\int_\Omega (\Theta_{i-1}^k - \Theta_i^k)\,dx \le \int_\Omega Q\psi_i^k(x,\nabla v_{n,k})\,dx.$$

On the other hand, since $\chi_n(x)W_1 + (1-\chi_n(x))W_i^k(x,\cdot) \in G_{\chi_n(x)+(1-\chi_n(x))\theta_i^k(x)}(W_1, W_{i-1}^k(x,\cdot))$, where

$$\theta_i^k = \frac{\Theta_{i-1}^k - \Theta_i^k}{\Theta_{i-1}^k},$$

we deduce from the expression (5.6) of $Q\psi_i^k$ that

$$\begin{aligned}
\int_\Omega W_i^k(x,\nabla u_i^k)\,dx + \kappa\int_\Omega (\Theta_{i-1}^k - \Theta_i^k)\,dx \;&\le\; \int_\Omega \Big[\chi_n W_1(\nabla v_{n,k}) + (1-\chi_n)W_i^k(x,\nabla v_{n,k})\Big]\,dx\\
&\qquad +\kappa\int_\Omega \Theta_{i-1}^k\big[\chi_n + (1-\chi_n)\theta_i^k\big]\\
&= \int_\Omega \Big[\chi_n W_1(\nabla v_{n,k}) + (1-\chi_n)W_i^k(x,\nabla v_{n,k})\Big]\,dx\\
&\qquad +\kappa\int_\Omega \big[\Theta_{i-1}^k\chi_n + (1-\chi_n)(\Theta_{i-1}^k - \Theta_i^k)\big].
\end{aligned}$$

Define $\Xi^k(t) = \Theta_{i-1}^k$ if $t \in [t_i^k, t_{i+1}^k)$ and $\Xi(t)$ its weak* limit in $L^\infty(\Omega;[0,1])$. Thanks to (5.39) we get by passing to the limit as $k \to +\infty$ that

$$\begin{aligned}
&\int_\Omega W(t)(x,\nabla u(t))\,dx + \kappa\int_\Omega (\Xi(t) - \Theta(t))\,dx\\
&\quad\le \int_\Omega \Big[\chi_n W_1(\nabla v_n) + (1-\chi_n)W(t)(x,\nabla v_n)\Big]\,dx\\
&\qquad +\kappa\int_\Omega \big[\Xi(t)\chi_n + (1-\chi_n)(\Xi(t)-\Theta(t))\big]\,dx.
\end{aligned}$$

Sending then $n \to +\infty$ yields, according to (5.38),

$$\int_\Omega W(t)(x,\nabla u(t))\,dx + \kappa\int_\Omega (\Xi(t)-\Theta(t))\,dx$$
$$\leq \int_\Omega W(x,\nabla v)\,dx + \kappa\int_\Omega \big[\Xi(t)\theta + (1-\theta)(\Xi(t)-\Theta(t))\big]\,dx,$$

and thus

$$\int_\Omega W(t)(x,\nabla u(t))\,dx \leq \int_\Omega W(x,\nabla v)\,dx + \kappa\int_\Omega \Theta(t)\theta\,dx,$$

which is exactly the desired unilateral minimality property. □

Finally, as a consequence of the fact that $W(t)(x,\cdot) \in G_{1-\Theta(t)(x)}(W_1,W_2)$ for a.e. $x\in\Omega$ and all $t\in[0,T]$, together with the local character of G-closure (4.3), we deduce that the quasistatic evolution obtained in Theorem 1.1 is not too low in the sense that the total energy is as close as we want from the one associated to the original model.

Proposition 5.5. *Let $(u(t),\Theta(t),W(t))$ be any quasistatic evolution given by Theorem 1.1. Then there exists a time dependent sequence of characteristic functions $(\chi_n(t)) \subset L^\infty(\Omega;\{0,1\})$ which is increasing with respect to t, such that for any $t\in[0,T]$,*

$$\begin{cases} \chi_n(t) \xrightharpoonup{*} 1-\Theta(t) \text{ in } L^\infty(\Omega;[0,1]), \\ \displaystyle\int_\Omega W_{\chi_n(t)}(x,\nabla v)\,dx \xrightarrow{\Gamma} \int_\Omega W(t)(x,\nabla v)\,dx, \end{cases}$$

and if $v_n(t)\in\mathcal{A}(t)$ is the (unique) solution of

$$\min_{v\in\mathcal{A}(t)} \int_\Omega W_{\chi_n(t)}(x,\nabla v)\,dx,$$

then $v_n(t) \rightharpoonup u(t)$ in $W^{1,p}(\Omega)$, and

$$\int_\Omega W_{\chi_n(t)}(x,\nabla v_n(t))\,dx \to \int_\Omega W(t)(x,\nabla u(t))\,dx.$$

6. Fracture versus damage

This section is devoted to the proof of Theorem 1.2 which is an existence result of a model of quasistatic evolution for a continuum that undergoes both damage and fracture. This study was initiated in [22]. Since the arguments will be very close to those employed in the previous section, we will not prove all the statements postponing to section 5 for more details.

In addition to the damage process, we assume that the material under consideration can experience fractures. The modeling of the fracture process is conceptually similar to that of damage (we refer to [30, 9] for detailed description of the model). At a discrete time level, and for a given time t_i, the created crack Γ_i will be assimilate to the union of the crack at the previous time step Γ_{i-1} and the jump set J_{u_i} of the current deformation field u_i which is solution of a minimization problem involving a surface energy which penalizes the presence of cracks. According to Griffith's theory we assume that the energy spent to produce a crack is proportional to its area so that the dissipative energy from the initial time up to time t_i is

$$G_c\,\mathcal{H}^{N-1}(\Gamma_i\setminus\partial_N\Omega),$$

where $G_c>0$ is the toughness of the material that we assume to be equal to 1 for simplicity. Note that there is no energy associated to the part of the crack that lies on $\partial_N\Omega$. On the other hand, it possible for the crack to reach the Dirichlet boundary $\partial_D\Omega$, and $\Gamma_i\cap\partial_D\Omega$ will be interpreted as those points of $\partial_D\Omega$ where the deformation u_i do not match the boundary datum $g(t_i)$.

For technical reasons, we assume now that the boundary datum $g\in W^{1,1}([0,T];W^{1,p}(\mathbb{R}^N))\cap L^\infty(\mathbb{R}^N\times[0,T])$. The reason why we further impose an L^∞ bound is that it will enable us to apply the maximum principle in the following minimization problems, ensuring their wellposedness.

Since $\partial_D\Omega$ is open in the relative topology of $\partial\Omega$, there exists an open set $\Omega'\subset\mathbb{R}^N$ such that $\Omega\subset\Omega'$ and $\Omega'\cap\partial\Omega=\partial_D\Omega$. Then the space of all kinematically admissible deformation fields is given by

$$\mathcal{A}(t) := \{v\in SBV^p(\Omega') : v = g(t) \text{ a.e. on } \Omega'\setminus\overline{\Omega}\}.$$

Note that if $v \in \mathcal{A}(t)$, then $v = g(t)$ $\mathcal{H}^{N-1}$-a.e. on $\partial_D\Omega \setminus J_v$, where we still denote by v the inner trace of v on $\partial_D\Omega$.

6.1. **Time discretization.**

6.1.1. *First time step.* At the initial time, $g_0^k = g(0)$, and one wants to minimize

$$(v,\chi) \mapsto \int_\Omega \Big[\chi W_1(\nabla v) + (1-\chi)W_2(\nabla v) + \kappa\chi\Big]\,dx + \mathcal{H}^{N-1}(J_v \setminus \partial_N\Omega),$$

among all $(v,\chi) \in \mathcal{A}(0) \times L^\infty(\Omega;\{0,1\})$. Minimizing first with respect to χ leads to the following nonconvex integrand

$$\psi_0(\xi) := \min\{W_1(\xi)+\kappa, W_2(\xi)\},$$

and the previous minimization problem is equivalent to

$$I_0 := \inf_{v\in\mathcal{A}(0)} \left\{\int_\Omega \psi_0(\nabla v)\,dx + \mathcal{H}^{N-1}(J_v \setminus \partial_N\Omega)\right\}. \tag{6.1}$$

The lack of convexity of the integrand ψ_0 may prevent (6.1) to have solutions, so that it is necessary to compute the relaxed problem. Since the boundary datum $g(0) \in L^\infty(\Omega')$, then by the maximum principle there is no loss of generality to assume that the minimizing sequences are uniformly bounded in $L^\infty(\Omega')$. Hence by Ambrosio's compactness Theorem (Theorem 2.1), the limit functional space remains $SBV^p(\Omega')$ and to get the relaxed functional, it suffices to replace ψ_0 by its convexification (see Theorem 3.4), which coincides in the scalar case of its quasiconvexification $Q\psi_0$ defined by

$$Q\psi_0(\xi) := \inf_{\varphi\in W^{1,p}_{\rm per}(Q)} \int_Q \psi_0(\xi+\nabla\varphi)\,dx,$$

or by Lemma 5.1

$$Q\psi_0(\xi) = \min_{\theta\in[0,1]}\left[\min_{W^*\in G_\theta(W_1,W_2)} W^*(\xi) + \kappa\theta\right].$$

Then by Theorem 3.4

$$I_0 = \min_{v\in\mathcal{A}(0)} \left\{\int_\Omega Q\psi_0(\nabla v)\,dx + \mathcal{H}^{N-1}(J_v \setminus \partial_N\Omega)\right\}. \tag{6.2}$$

Let $u_0 \in \mathcal{A}(0)$ be a minimizer of (6.2). By Lemmas 5.1 and 5.2, there exist $\theta_0 \in L^\infty(\Omega;[0,1])$ and a Carathéodory function $W_0 \in \mathcal{F}(\Omega,\alpha,\beta,p)$ such that for a.e. $x \in \Omega$, $W_0(x,\cdot) \in G_{\theta_0(x)}(W_1,W_2)$ is uniformly convex and of class $\mathcal{C}^1$, and

$$Q\psi_0(\nabla u_0(x)) = W_0(x,\nabla u_0(x)) + \kappa\theta_0(x).$$

We now define the total energy at the initial time by

$$\begin{aligned}\mathcal{E}_0 &:= \int_\Omega Q\psi_0(\nabla u_0)\,dx + \mathcal{H}^{N-1}(J_{u_0}\setminus\partial_N\Omega)\\ &= \int_\Omega W_0(x,\nabla u_0)\,dx + \kappa\int_\Omega (1-\Theta_0)\,dx + \mathcal{H}^{N-1}(\Gamma_0\setminus\partial_N\Omega) = I_0,\end{aligned}$$

where $\Theta_0 := 1-\theta_0$ is the local volume fraction of the undamaged material, and $\Gamma_0 := J_{u_0}$ is the crack created at the initial time.

6.1.2. *Subsequent time steps.* Let $i \geq 1$, and assume that there exist a Carathéodory function $W^k_{i-1} \in \mathcal{F}(\Omega,\alpha,\beta,p)$, $\Theta^k_{i-1} \in L^\infty(\Omega;[0,1])$ such that $W^k_{i-1}(x,\cdot) \in G_{1-\Theta^k_{i-1}(x)}(W_1,W_2)$ for a.e. $x \in \Omega$, and $\Gamma^k_{i-1} \in \mathcal{R}(\overline{\Omega})$. At time t_i^k one wants to minimize

$$(v,\chi) \mapsto \int_\Omega \Big[\chi W_1(\nabla v) + (1-\chi)W^k_{i-1}(x,\nabla v) + \kappa\chi\Big]\,dx + \kappa\int_\Omega \Theta^k_{i-1}\chi\,dx + \mathcal{H}^{N-1}(J_v \setminus (\Gamma^k_{i-1}\cup\partial_N\Omega)),$$

among all $(v,\chi) \in \mathcal{A}(t_i^k) \times L^\infty(\Omega;\{0,1\})$. We first minimize with respect to χ which leads to the following nonconvex Carathéodory integrand

$$\psi_i^k(x,\xi) := \min\{W_1(\xi) + \kappa\Theta^k_{i-1}(x), W^k_{i-1}(x,\xi)\},$$

and the previous minimization problem is equivalent to

$$I_i^k := \inf_{v\in\mathcal{A}(t_i^k)} \left\{ \int_\Omega \psi_i^k(x,\nabla v)\,dx + \mathcal{H}^{N-1}(J_v \setminus (\Gamma_{i-1}^k \cup \partial_N\Omega)) \right\}. \tag{6.3}$$

Once again, the lack of convexity of the integrand ψ_i^k may prevent (6.3) to have solutions, and it is necessary to compute the relaxed problem. As before it suffices to replace ψ_i^k by its (quasi)convexification defined by

$$Q\psi_i^k(x,\xi) := \inf_{\varphi\in W^{1,p}_{\rm per}(Q)} \int_Q \psi_i^k(x,\xi+\nabla\varphi(y))\,dy,$$

or still, by Lemma 5.1,

$$Q\psi_i^k(x,\xi) = \min_{\theta\in[0,1]} \left[\min_{W^*\in G_\theta(W_1,W_{i-1}^k(x,\cdot))} W^*(\xi) + \kappa\Theta_{i-1}^k(x)\theta \right]. \tag{6.4}$$

Then, by Theorem 3.4

$$I_i^k = \min_{v\in\mathcal{A}(t_i^k)} \left\{ \int_\Omega Q\psi_i^k(x,\nabla v)\,dx + \mathcal{H}^{N-1}(J_v \setminus (\Gamma_{i-1}^k \cup \partial_N\Omega)) \right\}. \tag{6.5}$$

Let $u_i^k \in \mathcal{A}(t_i^k)$ be a solution of (6.5), *i.e.*,

$$I_i^k = \int_\Omega Q\psi_i^k(x,\nabla u_i^k)\,dx + \mathcal{H}^{N-1}(J_{u_i^k} \setminus (\Gamma_{i-1}^k \cup \partial_N\Omega)).$$

By Lemma 5.3, there exist $\theta_i^k \in L^\infty(\Omega;[0,1])$ and a Carathéodory function $W_i^k \in \mathcal{F}(\Omega,\alpha,\beta,p)$ such that for a.e. $x\in\Omega$, $W_i^k(x,\cdot) \in G_{\theta_i^k(x)}(W_1,W_{i-1}^k(x,\cdot))$ is uniformly convex and of class $\mathcal{C}^1$, and

$$Q\psi_i^k(x,\nabla u_i^k(x)) = W_i^k(x,\nabla u_i^k(x)) + \kappa\Theta_{i-1}^k(x)\theta_i^k(x).$$

We define the volume fraction of the undamaged material and the crack at time t_i^k by

$$\Theta_i^k := \Theta_{i-1}^k(1-\theta_i^k) \quad \text{and} \quad \Gamma_i^k := \Gamma_{i-1}^k \cup J_{u_i^k}.$$

Then, as in the proof of Lemma 5.3 one has $W_i^k(x,\cdot) \in G_{1-\Theta_i^k(x)}(W_1,W_2)$ for a.e. $x\in\Omega$, and

$$I_i^k = \int_\Omega W_i^k(x,\nabla u_i^k)\,dx + \kappa\int_\Omega (\Theta_{i-1}^k - \Theta_i^k)\,dx + \mathcal{H}^{N-1}(J_{u_i^k} \setminus (\Gamma_{i-1}^k \cup \partial_N\Omega)). \tag{6.6}$$

We also define the total energy at time t_i^k by

$$\begin{aligned} \mathcal{E}_i^k &:= I_i^k + \kappa\int_\Omega (1-\Theta_{i-1}^k)\,dx + \mathcal{H}^{N-1}(\Gamma_{i-1}^k \setminus \partial_N\Omega) \\ &= \int_\Omega W_i^k(x,\nabla u_i^k)\,dx + \kappa\int_\Omega (1-\Theta_i^k)\,dx + \mathcal{H}^{N-1}(\Gamma_i^k \setminus \partial_N\Omega). \end{aligned}$$

6.2. **A few properties of the discrete evolution.** Since u_i^k is a solution of (6.5) and $W_i^k(x,\cdot) \in G_{\theta_i^k(x)}(W_1,W_{i-1}^k(x,\cdot))$ for a.e. $x\in\Omega$, it follows that u_i^k is also a solution of

$$\inf_{v\in\mathcal{A}(t_i^k)} \left\{ \int_\Omega W_i^k(x,\nabla v)\,dx + \mathcal{H}^{N-1}(J_v \setminus (\Gamma_{i-1}^k \cup \partial_N\Omega)) \right\}, \tag{6.7}$$

and the pair (u_i^k,Γ_i^k) satisfies the following unilateral minimality property:

$$\int_\Omega W_i^k(x,\nabla u_i^k)\,dx + \mathcal{H}^{N-1}(\Gamma_i^k \setminus \partial_N\Omega) \le \int_\Omega W_i^k(x,\nabla v)\,dx + \mathcal{H}^{N-1}(K \setminus \partial_N\Omega), \tag{6.8}$$

for any $K \in \mathcal{R}(\overline{\Omega})$ satisfying $\Gamma_i^k \,\tilde{\subset}\, K$, and any $v\in\mathcal{A}(t_i^k)$ with $J_v \,\tilde{\subset}\, K$.

Moreover, W_i^k satisfies the same monotonicity properties than in Lemmas 5.4 and 5.5, and arguing exactly as in section 5.3.3, we derive the following upper bound on the total energy:

$$\mathcal{E}_j^k \le \mathcal{E}_0 + \sum_{i=0}^{j-1} \int_\Omega DW_i^k(x,\nabla u_i^k + s_i^k(\nabla g_{i+1}^k - \nabla g_i^k)) \cdot (\nabla g_{i+1}^k - \nabla g_i^k)\,dx. \tag{6.9}$$

6.3. **Piecewise constant interpolation.** For each $t \in [t_i^k, t_{i+1}^k)$, we set

$$g_k(t) := g_i^k, \quad u_k(t) := u_i^k, \quad \Theta_k(t) := \Theta_i^k, \quad \Gamma_k(t) := \Gamma_i^k, \quad W_k(t) := W_i^k,$$

and we define $\mathcal{A}_k(t) := \mathcal{A}(t_i^k)$ and $\mathcal{E}_k(t) := \mathcal{E}_i^k$. In particular $u_k(t) \in \mathcal{A}_k(t)$, and by (6.8) the pair $(u_k(t), \Gamma_k(t))$ satisfies the following unilateral minimality property: for every $t \in [0,T]$,

$$\int_\Omega W_k(t)(x, \nabla u_k(t))\, dx + \mathcal{H}^{N-1}(\Gamma_k(t) \setminus \partial_N\Omega) \leq \int_\Omega W_k(t)(x, \nabla v)\, dx + \mathcal{H}^{N-1}(K \setminus \partial_N \Omega), \qquad (6.10)$$

for any $K \in \mathcal{R}(\overline{\Omega})$ satisfying $\Gamma_k(t) \,\widetilde{\subset}\, K$, and any $v \in \mathcal{A}_k(t)$ with $J_v \,\widetilde{\subset}\, K$. Moreover, the mappings $t \mapsto W_k(t) \in \mathcal{F}(\Omega, \alpha, \beta, p)$ and $t \mapsto \Theta_k(t) \in L^\infty(\Omega; [0,1])$ are decreasing, and $t \mapsto \Gamma_k(t)$ is increasing for every $k \in \mathbb{N}$.

As in the previous section, there exists a Bochner integrable function $\Phi_k : [0,T] \to L^p(\Omega; \mathbb{R}^N)$ satisfying $\|\Phi_k(t)\|_{L^p(\Omega;\mathbb{R}^N)} \to 0$ uniformly with respect to $t \in [0,T]$, such that the energy estimate (6.9) can be rewritten as

$$\mathcal{E}_k(t) \leq \mathcal{E}_0 + \int_0^t \int_\Omega DW_k(\tau)(x, \nabla u_k(\tau) + \Phi_k(\tau)) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau. \qquad (6.11)$$

6.4. **The time continuous limit.** We now pass to the limit as the time step tends to zero. We first state a compactness result on $(u_k(t), \Theta_k(t), \Gamma_k(t), W_k(t))$.

Proposition 6.1. *There exist a subsequence (not relabeled), a set $\Gamma(t) \in \mathcal{R}(\overline{\Omega})$, and functions $u(t) \in \mathcal{A}(t)$, $\Theta(t) \in L^\infty(\Omega;[0,1])$ and $W(t) \in \mathcal{F}(\Omega, \alpha, \beta, p)$ such that*

$$\begin{cases} \Gamma_k(t) \ \sigma\text{-converges to } \Gamma(t), \\ u_k(t) \rightharpoonup u(t) \ \text{in } SBV^p(\Omega), \\ \Theta_k(t) \xrightharpoonup{*} \Theta(t) \ \text{in } L^\infty(\Omega;[0,1]), \\ \displaystyle\int_\Omega W(t)(x, \nabla v)\, dx = \Gamma\text{-}\lim_{k \to +\infty} \int_\Omega W_k(t)(x, \nabla v)\, dx. \end{cases}$$

Moreover the map $(u, \nabla u) : [0,T] \to L^p(\Omega) \times L^p(\Omega;\mathbb{R}^N)$ is strongly measurable, the maps $t \mapsto \Theta(t)$ and $t \mapsto W(t)$ are decreasing and $t \mapsto \Gamma(t)$ is increasing, and $W(t)(x,\cdot) \in G_{1-\Theta(t)(x)}(W_1, W_2)$ is uniformly convex and of class $\mathcal{C}^1$ for a.e. $x \in \Omega$ and every $t \in [0,T]$. Finally, we have convergence of the bulk energy at every time $t \in [0,T]$,

$$\int_\Omega W(t)(x, \nabla u(t))\, dx = \lim_{k \to +\infty} \int_\Omega W_k(t)(x, \nabla u_k(t))\, dx. \qquad (6.12)$$

Proof. The compactness of the pair $(\Theta_k(t), W_k(t))$ and the properties of the limit $(\Theta(t), W(t))$ can be obtained exactly as in the proof of Proposition 5.1.

We now deal with the compactness of $(\Gamma_k(t))$ and $(u_k(t))$ by first deriving *a priori* estimates. From the maximum principle, we have that $\|u_i^k\|_{L^\infty(\Omega')} \leq \|g_i^k\|_{L^\infty(\Omega')}$ and thus

$$\sup_{t \in [0,T]} \sup_{k \in \mathbb{N}} \|u_k(t)\|_{L^\infty(\Omega')} < +\infty. \qquad (6.13)$$

Then taking $v = g_i^k$ as test function in (6.7), we get from the p-growth and p-coercivity conditions (2.1) satisfied by $W_k(t)$ that $\alpha \|\nabla u_i^k\|^p_{L^p(\Omega;\mathbb{R}^N)} \leq \beta\big(1 + \|\nabla g_i^k\|^p_{L^p(\Omega;\mathbb{R}^N)}\big)$, and thus

$$\sup_{t \in [0,T]} \sup_{k \in \mathbb{N}} \|\nabla u_k(t)\|_{L^p(\Omega';\mathbb{R}^N)} < +\infty. \qquad (6.14)$$

Finally from the energy estimate (6.11), the $(p-1)$-growth condition (2.3) satisfied by $DW_k(t)$, Hölder's inequality and (6.14), we get that

$$\sup_{t \in [0,T]} \sup_{k \in \mathbb{N}} \mathcal{H}^{N-1}(\Gamma_k(t)) < +\infty. \qquad (6.15)$$

According to (6.15), Proposition 3.1 and a variant of Helly's Theorem for increasing set functions (see *e.g.* [18, Theorem 6.3] for the case of Hausdorff converging compact sets), one can assume that for the same subsequence, $\Gamma_k(t)$ σ-converges to some $\Gamma(t) \in \mathcal{R}(\overline{\Omega})$ which is still increasing with respect to t.

Moreover, according to (6.13), (6.14), (6.15) and the fact that (by construction) $J_{u_k(t)} \subset \Gamma_k(t)$, we deduce that

$$\sup_{t\in[0,T]} \sup_{k\in\mathbb{N}} \left(\|u_k(t)\|_{L^\infty(\Omega')} + \|\nabla u_k(t)\|_{L^p(\Omega';\mathbb{R}^N)} + \mathcal{H}^{N-1}(J_{u_k(t)})\right) < +\infty. \tag{6.16}$$

Hence by (6.16) and Ambrosio's compactness Theorem (Theorem 2.1), for each $t \in [0,T]$, there exist a t-dependent subsequence (k_j) and $u(t) \in SBV^p(\Omega')$ such that $u_{k_j}(t) \rightharpoonup u(t)$ in $SBV^p(\Omega')$. Moreover, the fact that $g_k(t) \to g(t)$ strongly in $W^{1,p}(\Omega')$ for every $t \in [0,T]$ ensures that $u(t) = g(t)$ a.e. on $\Omega' \setminus \overline{\Omega}$, so that $u(t) \in \mathcal{A}(t)$. By (6.10), Lemma 5.6 and Theorem 3.5, we deduce that the pair $(u(t), \Gamma(t))$ is a unilateral minimizer with respect to the integrand $W(t)$, *i.e.*,

$$\int_\Omega W(t)(x, \nabla u(t))\, dx + \mathcal{H}^{N-1}(\Gamma(t) \setminus \partial_N \Omega) \le \int_\Omega W(t)(x, \nabla v)\, dx + \mathcal{H}^{N-1}(K \setminus \partial_N \Omega),$$

for any $K \in \mathcal{R}(\overline{\Omega})$ satisfying $\Gamma(t) \,\tilde{\subset}\, K$, and any $v \in \mathcal{A}(t)$ satisfying $J_v \,\tilde{\subset}\, K$. In particular, if we fix $K = \Gamma(t)$, since $W(t)(x, \cdot)$ is in particular strictly convex (see the proof of Proposition 5.1), then the minimization problem

$$\min \left\{ \int_\Omega W(t)(x, \nabla v)\, dx : v \in \mathcal{A}(t),\ J_v \,\tilde{\subset}\, \Gamma(t) \right\}$$

admits a unique solution which must be $u(t)$. Hence by uniqueness the whole sequence $u_k(t)$ weakly converges to $u(t)$ in $SBV^p(\Omega)$ and using again Theorem 3.5, we infer that the convergence of the bulk energy (6.12) holds. Finally, one can show exactly as in the proof of Proposition 5.1 that the map $(u, \nabla u) : [0,T] \to L^p(\Omega) \times L^p(\Omega; \mathbb{R}^N)$ is strongly measurable. $\square$

We define the total energy at time $t \in [0,T]$ by

$$\mathcal{E}(t) := \int_\Omega W(t)(x, \nabla u(t))\, dx + \kappa \int_\Omega (1 - \Theta(t))\, dx + \mathcal{H}^{N-1}(\Gamma(t) \setminus \partial_N \Omega).$$

Our next goal is to prove the energy balance. We start by proving the upper bound inequality on the total energy.

Proposition 6.2. *For any $t \in [0,T]$, one has*

$$\mathcal{E}(t) \le \mathcal{E}_0 + \int_0^t \int_\Omega DW(\tau)(x, \nabla u(\tau)) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau. \tag{6.17}$$

Proof. By the Γ-convergence result (5.16) and Theorem 3.4, we have that

$$\mathcal{E}(t) \le \liminf_{k\to+\infty} \left\{ \int_\Omega W_k(t)(x, \nabla u_k(t))\, dx + \kappa \int_\Omega (1 - \Theta_k(t))\, dx + \mathcal{H}^{N-1}(\Gamma_k(t) \setminus \partial_N \Omega) \right\}, \tag{6.18}$$

and from the upper bound estimate (6.11), we get that

$$\mathcal{E}(t) \le \mathcal{E}_0 + \limsup_{k\to+\infty} \int_0^t \int_\Omega DW_k(\tau)(x, \nabla u_k(\tau) + \Phi_k(\tau)) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau. \tag{6.19}$$

Thanks to (6.12) and arguing exactly as in the proof of Proposition 5.2 we conclude that

$$\begin{aligned}
\lim_{k\to+\infty} &\int_0^t \int_\Omega DW_k(\tau)(x, \nabla u_k(\tau) + \Phi_k(\tau)) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau \\
&= \lim_{k\to+\infty} \int_0^t \int_\Omega DW_k(\tau)(x, \nabla u_k(\tau)) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau \\
&= \int_0^t \int_\Omega DW(\tau)(x, \nabla u(\tau)) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau,
\end{aligned} \tag{6.20}$$

which, together with (6.19) completes the proof of (6.17). $\square$

We next prove that the energy inequality (6.17) is actually an equality. Following [17] the argument rests on the approximation of the Bochner integral by suitable Riemann sums (see Lemma 5.7).

Proposition 6.3. *For any $t \in [0,T]$, then*

$$\mathcal{E}(t) \geq \mathcal{E}_0 + \int_0^t \int_\Omega DW(\tau)(x, \nabla u(\tau)) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau.$$

Moreover, for every $t \in [0,T]$, there is convergence of the surface energy

$$\mathcal{H}^{N-1}(\Gamma_k(t) \setminus \partial_N \Omega) \to \mathcal{H}^{N-1}(\Gamma(t) \setminus \partial_N \Omega)$$

as well as the total energy $\mathcal{E}_k(t) \to \mathcal{E}(t)$.

Proof. Let $s < t$. By the Jump Transfer Theorem (Theorem 3.6), since $u(t) + g(s) - g(t) \in \mathcal{A}(s)$, one can find a sequence (v_k) such that $v_k \in \mathcal{A}_k(s)$ for each $k \in \mathbb{N}$, $v_k \rightharpoonup u(t) + g(s) - g(t)$ in $SBV^p(\Omega')$,

$$\lim_{k \to +\infty} \int_\Omega W_k(t)(x, \nabla v_k)\, dx = \int_\Omega W(t)(x, \nabla u(t) + \nabla g(s) - \nabla g(t))\, dx, \tag{6.21}$$

and

$$\limsup_{k \to +\infty} \mathcal{H}^{N-1}(J_{v_k} \setminus (\Gamma_k(s) \cup \partial_N \Omega)) \leq \mathcal{H}^{N-1}(J_{u(t)} \setminus (\Gamma(s) \cup \partial_N \Omega)). \tag{6.22}$$

Let $i \leq j$ be such that $s \in [t_i^k, t_{i+1}^k)$ and $t \in [t_j^k, t_{j+1}^k)$. Since by Lemma 5.5

$$W_j^k(x, \cdot) \in G_{1 - \frac{\Theta_j^k(x)}{\Theta_{i-1}^k(x)}}(W_1, W_{i-1}^k(x, \cdot))$$

for a.e. $x \in \Omega$, then from (6.4), (6.5) and (6.6) we have that

$$\begin{aligned} &\int_\Omega W_i^k(x, \nabla u_i^k)\, dx + \kappa \int_\Omega (\Theta_{i-1}^k - \Theta_i^k)\, dx + \mathcal{H}^{N-1}(J_{u_i^k} \setminus (\Gamma_{i-1}^k \cup \partial_N \Omega)) \\ &\qquad \leq \int_\Omega W_j^k(x, \nabla v_k)\, dx + \kappa \int_\Omega \Theta_{i-1}^k \left(1 - \frac{\Theta_j^k(x)}{\Theta_{i-1}^k(x)} \right) dx + \mathcal{H}^{N-1}(J_{v_k} \setminus (\Gamma_{i-1}^k \cup \partial_N \Omega)), \end{aligned}$$

and thus

$$\begin{aligned} &\int_\Omega W_k(s)(x, \nabla u_k(s))\, dx + \kappa \int_\Omega (1 - \Theta_k(s))\, dx \\ &\qquad \leq \int_\Omega W_k(t)(x, \nabla v_k)\, dx + \kappa \int_\Omega (1 - \Theta_k(t))\, dx + \mathcal{H}^{N-1}(J_{v_k} \setminus (\Gamma_k(s) \cup \partial_N \Omega)). \end{aligned}$$

Hence taking the limit as $k \to +\infty$, (6.21) and (6.22) lead to

$$\begin{aligned} &\int_\Omega W(s)(x, \nabla u(s))\, dx + \kappa \int_\Omega (1 - \Theta(s)) dx \\ &\quad \leq \int_\Omega W(t)(x, \nabla u(t) + \nabla g(s) - \nabla g(t))\, dx + \kappa \int_\Omega (1 - \Theta(t)) dx + \mathcal{H}^{N-1}(J_{u(t)} \setminus (\Gamma(s) \cup \partial_N \Omega)), \end{aligned}$$

and consequently, since $J_{u(t)} \,\tilde{\subset}\, \Gamma(t)$,

$$\mathcal{E}(s) \leq \mathcal{E}(t) + \int_\Omega W(t)(x, \nabla u(t) + \nabla g(s) - \nabla g(t))\, dx - \int_\Omega W(t)(x, \nabla u(t))\, dx.$$

The rest of the proof of the energy inequality is exactly the same than that of Proposition 5.3 by choosing a suitable subdivision $(s_i^k)_{0 \leq i \leq i_k}$ of the interval $[0,t]$ thanks to Lemma 5.7.

To prove the convergence of the total energy, for $t \in [0,T]$, we have thanks to (6.18) and (6.20)

$$\begin{aligned} \mathcal{E}(t) &\leq \liminf_{k \to +\infty} \mathcal{E}_k(t) \leq \limsup_{k \to +\infty} \mathcal{E}_k(t) \\ &\leq \mathcal{E}_0 + \limsup_{k \to +\infty} \int_0^t \int_\Omega DW_k(\tau)(x, \nabla u_k(\tau)) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau \\ &= \mathcal{E}_0 + \int_0^t \int_\Omega DW(\tau)(x, \nabla u(\tau)) \cdot \nabla \dot{g}(\tau)\, dx\, d\tau \leq \mathcal{E}(t). \end{aligned}$$

Then the convergence of the surface energy follows as a consequence of (6.12). □

We next prove that $(u(t), \Theta(t), \Gamma(t), W(t))$ satisfies a unilateral minimality property.

Proposition 6.4. *Let $t \in [0,T]$. For any $K \in \mathcal{R}(\overline{\Omega})$ such that $\Gamma(t) \,\widetilde{\subset}\, K$, any $v \in \mathcal{A}(t)$ such that $J_v \,\widetilde{\subset}\, K$, any $\theta \in L^\infty(\Omega;[0,1])$ and any $W \in \mathcal{F}(\Omega,\alpha,\beta,p)$ such that $W(x,\cdot) \in G_{\theta(x)}(W_1, W(t)(x,\cdot))$ for a.e. $x \in \Omega$, then*

$$\int_\Omega W(t)(x,\nabla u(t))\,dx + \mathcal{H}^{N-1}(\Gamma(t)\setminus\partial_N\Omega) \le \int_\Omega W(x,\nabla v)\,dx + \mathcal{H}^{N-1}(K\setminus\partial_N\Omega) + \kappa\int_\Omega \Theta(t)\theta\,dx.$$

Proof. Let $t \in [0,T]$ and take v, K, W and θ as above. By Proposition 4.1 there exists a sequence of characteristic functions $(\chi_n) \subset L^\infty(\Omega;\{0,1\})$ such that $\chi_n \xrightharpoonup{*} \theta$ in $L^\infty(\Omega;[0,1])$, and

$$\int_\Omega W(x,\nabla v)\,dx = \Gamma\text{-}\lim_{n\to+\infty}\int_\Omega \Big[\chi_n W_1(\nabla v) + (1-\chi_n)W(t)(x,\nabla v)\Big]\,dx.$$

Moreover, since

$$\int_\Omega W(t)(x,\nabla v)\,dx = \Gamma\text{-}\lim_{k\to+\infty}\int_\Omega W_k(t)(x,\nabla v)\,dx,$$

we deduce from Lemma 4.1 that for each $n \in \mathbb{N}$,

$$\int_\Omega \Big[\chi_n W_1(\nabla v) + (1-\chi_n)W(t)(x,\nabla v)\Big]\,dx = \Gamma\text{-}\lim_{k\to+\infty}\int_\Omega \Big[\chi_n W_1(\nabla v) + (1-\chi_n)W_k(t)(x,\nabla v)\Big]\,dx.$$

According to Theorem 3.4 and Lemma 3.3, there exists a recovery sequence $(v_n) \subset SBV^p(\Omega)$ weakly converging to v in $SBV^p(\Omega)$ such that $v_n \in \mathcal{A}(t)$ for each $n \in \mathbb{N}$ and

$$\begin{aligned}\int_\Omega & W(x,\nabla v)\,dx + \mathcal{H}^{N-1}(J_v\setminus(\Gamma(t)\cup\partial_N\Omega))\\ &= \lim_{n\to+\infty}\left(\int_\Omega \Big[\chi_n W_1(\nabla v_n) + (1-\chi_n)W(t)(x,\nabla v_n)\Big]\,dx + \mathcal{H}^{N-1}(J_{v_n}\setminus(\Gamma(t)\cup\partial_N\Omega))\right).\end{aligned} \tag{6.23}$$

By the Jump Transfer Theorem (Theorem 3.6), we can consider a sequence $(v_{n,k})_k \subset SBV^p(\Omega')$ such that $v_{n,k} \rightharpoonup v_n$ in $SBV^p(\Omega')$ as $k \to +\infty$, $v_{n,k} \in \mathcal{A}_k(t)$ for each $k \in \mathbb{N}$,

$$\begin{aligned}\int_\Omega & \Big[\chi_n W_1(\nabla v_n) + (1-\chi_n)W(t)(x,\nabla v_n)\Big]\,dx\\ &= \lim_{k\to+\infty}\int_\Omega \Big[\chi_n W_1(\nabla v_{n,k}) + (1-\chi_n)W_k(t)(x,\nabla v_{n,k})\Big]\,dx,\end{aligned} \tag{6.24}$$

and

$$\limsup_{k\to+\infty}\mathcal{H}^{N-1}(J_{v_{n,k}}\setminus(\Gamma_k(t)\cup\partial_N\Omega)) \le \mathcal{H}^{N-1}(J_{v_n}\setminus(\Gamma(t)\cup\partial_N\Omega)). \tag{6.25}$$

Fix $k \in \mathbb{N}$ and let i be such that $t \in [t_i^k, t_{i+1}^k)$, by (6.5) and (6.6), we have that

$$\begin{aligned}\int_\Omega W_i^k(x,\nabla u_i^k)\,dx &+ \kappa\int_\Omega(\Theta_{i-1}^k - \Theta_i^k)\,dx + \mathcal{H}^{N-1}(J_{u_i^k}\setminus(\Gamma_{i-1}^k\cup\partial_N\Omega))\\ &\le \int_\Omega Q\psi_i^k(x,\nabla v_{n,k})\,dx + \mathcal{H}^{N-1}(J_{v_{n,k}}\setminus(\Gamma_{i-1}^k\cup\partial_N\Omega)).\end{aligned}$$

On the other hand, since $\chi_n(x)W_1 + (1-\chi_n(x))W_i^k(x,\cdot) \in G_{\chi_n(x)+(1-\chi_n(x))\theta_i^k(x)}(W_1, W_{i-1}^k(x,\cdot))$, where

$$\theta_i^k = \frac{\Theta_{i-1}^k - \Theta_i^k}{\Theta_{i-1}^k},$$

we deduce from the expression (6.4) of $Q\psi_i^k$ that

$$\begin{aligned}\int_\Omega W_i^k(x,\nabla u_i^k)\,dx + \kappa\int_\Omega(\Theta_{i-1}^k - \Theta_i^k)\,dx &\le \int_\Omega \Big[\chi_n W_1(\nabla v_{n,k}) + (1-\chi_n)W_i^k(x,\nabla v_{n,k})\Big]\,dx\\ &\quad + \kappa\int_\Omega \Theta_{i-1}^k\big[\chi_n + (1-\chi_n)\theta_i^k\big]\,dx + \mathcal{H}^{N-1}(J_{v_{n,k}}\setminus(\Gamma_i^k\cup\partial_N\Omega))\\ &= \int_\Omega \Big[\chi_n W_1(\nabla v_{n,k}) + (1-\chi_n)W_i^k(x,\nabla v_{n,k})\Big]\,dx + \kappa\int_\Omega \big[\Theta_{i-1}^k\chi_n + (1-\chi_n)(\Theta_{i-1}^k - \Theta_i^k)\big]\,dx\\ &\quad + \mathcal{H}^{N-1}(J_{v_{n,k}}\setminus(\Gamma_i^k\cup\partial_N\Omega)).\end{aligned}$$

Define $\Xi^k(t) = \Theta^k_{i-1}$ if $t \in [t^k_i, t^k_{i+1})$ and $\Xi(t)$ its weak* limit in $L^\infty(\Omega;[0,1])$. Then

$$\int_\Omega W_k(t)(x,\nabla u_k(t))\,dx + \kappa\int_\Omega (\Xi_k(t)-\Theta_k(t))\,dx \leq \int_\Omega \Big[\chi_n W_1(\nabla v_{n,k}) + (1-\chi_n) W_k(t)(x,\nabla v_{n,k})\Big]\,dx$$
$$+\kappa\int_\Omega \big[\Xi_k(t)\chi_n + (1-\chi_n)(\Xi_k(t)-\Theta_k(t))\big]\,dx + \mathcal{H}^{N-1}(J_{v_{n,k}} \setminus (\Gamma_k(t)\cup\partial_N\Omega)),$$

and thanks to (6.24) and (6.25) we get by passing to the limit as $k \to +\infty$ that

$$\int_\Omega W(t)(x,\nabla u(t))\,dx + \kappa\int_\Omega (\Xi(t)-\Theta(t))\,dx \leq \int_\Omega \Big[\chi_n W_1(\nabla v_n) + (1-\chi_n) W(t)(x,\nabla v_n)\Big]\,dx$$
$$+\kappa\int_\Omega \big[\Xi(t)\chi_n + (1-\chi_n)(\Xi(t)-\Theta(t))\big]\,dx + \mathcal{H}^{N-1}(J_{v_n} \setminus (\Gamma(t)\cup\partial_N\Omega)).$$

Sending then $n \to +\infty$ yields, according to (6.23),

$$\int_\Omega W(t)(x,\nabla u(t))\,dx + \kappa\int_\Omega (\Xi(t)-\Theta(t))\,dx$$
$$\leq \int_\Omega W(x,\nabla v)\,dx + \kappa\int_\Omega \big[\Xi(t)\theta + (1-\theta)(\Xi(t)-\Theta(t))\big]\,dx + \mathcal{H}^{N-1}(J_v \setminus (\Gamma(t)\cup\partial_N\Omega)),$$

and thus since $J_v \,\widetilde{\subset}\, K$,

$$\int_\Omega W(t)(x,\nabla u(t))\,dx + \mathcal{H}^{N-1}(\Gamma(t)\setminus\partial_N\Omega) \leq \int_\Omega W(x,\nabla v)\,dx + \kappa\int_\Omega \Theta(t)\theta\,dx + \mathcal{H}^{N-1}(K\setminus\partial_N\Omega),$$

which is exactly the desired unilateral minimality property. □

Finally, using the fact that $W(t)(x,\cdot) \in G_{1-\Theta(t)(x)}(W_1,W_2)$ for a.e. $x \in \Omega$ and all $t \in [0,T]$ together with the local character of G-closure (4.3) and the Jump Transfer Theorem (Theorem 3.6), we deduce that the obtained quasistatic evolution is not too low in the sense that the elastic energy is as close as we want from the one associated to the original model.

Proposition 6.5. *Let $(u(t),\Theta(t),\Gamma(t),W(t))$ be any quasistatic evolution given by Theorem 1.2. Then there exists a time dependent sequence of characteristic functions $(\chi_n(t)) \subset L^\infty(\Omega;\{0,1\})$ which is increasing with respect to t, such that for any $t\in[0,T]$,*

$$\begin{cases} \chi_n(t) \overset{*}{\rightharpoonup} 1-\Theta(t) \text{ in } L^\infty(\Omega;[0,1]),\\ \displaystyle\int_\Omega W_{\chi_n(t)}(x,\nabla v)\,dx \xrightarrow{\Gamma} \int_\Omega W(t)(x,\nabla v)\,dx, \end{cases}$$

and if $v_n(t)\in\mathcal{A}(t)$ is the (unique) solution of

$$\min\left\{\int_\Omega W_{\chi_n(t)}(x,\nabla v)\,dx : \ v\in\mathcal{A}(t) \text{ and } J_v \,\widetilde{\subset}\, \Gamma(t)\right\},$$

then $v_n(t) \rightharpoonup u(t)$ in $SBV^p(\Omega)$, and

$$\int_\Omega W_{\chi_n(t)}(x,\nabla v_n(t))\,dx \to \int_\Omega W(t)(x,\nabla u(t))\,dx.$$

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